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ON THE DETERMINATION OF PLANAR GRAPHS

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THESIS

ON THE DETERMINATION OF PLANAR GRAPHS

by

Chong-hwa Kim

December 1974

Thesis Advisor: S.G. Chan

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proved that the algorithm can be applied to an arbitrary graph. Using this proposed algorithm, the rate of convergence of the algorithm is increased and the computer storage requirement is minimized.

On the Determination of Planar Graphs

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TABLE OF CONTENTS

I. INTRODUCTION ---------------------------------- 8

II. REVIEW OF EXISTING ALGORITHMS FOR TESTING THE PLANARITY OF A GRAPH ----------- 10

- A. FISHER-WING ALGORITHM ------------------ 10
  - 1. Decomposition Theorem -------------- 10
  - 2. The Planarity of a Pseudo-Hamiltonian Graph and the Attachment Matrix ------------------ 11
  - 3. Identification of a Planar Subgraph of an Arbitrary Graph ----- 13
    - a. Formation of the Decomposed Subgraphs --------------------- 13
    - b. Determination of C(k) ---------- 14
  - 4. Summary of the Iterative Algorithm in Terms of the Incidence Matrix --- 15
- B. LIN ALGORITHM -------------------------- 16
  - 1. Generalization of Euler's Theorem -- 16
  - 2. Graphs with a Hamiltonian Circuit -- 17
  - 3. The General Case ------------------ 18
- C. PHUNG-CHAN ALGORITHM ------------------ 20
  - 1. The Edge T-Matrix and Its Properties -------------------------- 20
  - 2. Planarity of a Pseudo-Hamiltonian Graph from its Edge T-Matrix ------- 23
  - 3. Determination of a Planar Subgraph of an Arbitrary Graph ------------- 24
    - a. Determination of C(k) ---------- 25
    - b. Determination of the Edge T-Matrix Associated with G'(k)-- 27

            c. Determination of the Edge T-Matrix Associated with a Decomposed Subgraph G(k+m) ----- 28

        4. Algorithm for Identifying a Planar Subgraph from an Arbitrary Graph --- 28

III. MODIFCATION OF PHUNG-CHAN ALGORITHM -------- 30

    A. ILLUSTRATION OF PHUNG-CHAN ALGORITHM --- 30

    B. THE PLANARITY OF A PSEUDO-HAMILTONIAN GRAPH ---------------------------------- 51

        1. Algorithm for Determining a Proper Pseudo-Euler Tree of G'(k) --------- 52

        2. Partition of Chord-Sets ------------ 53

        3. Identification of a Planar Subgraph of a Nonplanar Pseudo-Hamiltonian Graph ------------------------------ 55

    C. IDENTIFICATION OF A PLANAR SUBGRAPH OF AN ARBITRARY GRAPH ------------------ 56

        1. Algorithm for Finding C(k) of G(k) - 58

    D. ALGORITHM FOR IDENTIFYING A PLANAR SUBGRAPH FROM AN ARBITRARY GRAPH ------- 59

IV. APPLICATION OF ALGORITHM ------------------- 61

V. DISCUSSION ---------------------------------- 76

VI. CONCLUSION ---------------------------------- 78

LIST OF REFERENCES ---------------------------------- 79

INITIAL DISTRIBUTION LIST ---------------------------- 80

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I. INTRODUCTION

Recently, in the automated design of printed circuits for large scale digital computers, the search for an effective algorithm to test the planarity of a graph and to recognize a planar subgraph of a nonplanar graph have become important. The algorithm should be suitable for implementation on a digital computer and be practical for a graph which may have a large number of vertices and edges. And so the well known classical criteria of Kuratowsky [1] and Whitney [2] are not practical for the problems at hand.

The Auslander-Parter algorithm [3] tests the planarity of a graph by a decomposition procedure. This procedure breaks a large graph into several smaller graphs and finally the smaller graphs are tested by inspection. Goldstein algorithm [4] tests the planarity of a graph by inductively constructing the "meshes" of the graph. Fisher-Wing [5] and Lin [6] used decomposition theorem in their algorithms. Fisher-Wing introduced the concept of pseudo-Hamiltonian graph by using the decomposition theorem and reduced the problem of testing the planarity of a graph to the problem of testing the planarity of a set of pseudo-Hamiltonian graphs and Lin algorithm tests the planarity of a graph by repeatedly reducing a problem to several problems each of which involves a graph containing fewer edges than the graph of the original problem. Phung-Chan [7] improved the

Fisher-Wing algorithm by using the edge T-matrix associated with a graph instead of its incidence matrix which Fisher-Wing used in their algorithm.

In this paper the Phung-Chan algorithm is improved by modifying the method of application of the necessary and sufficient condition that a pseudo-Hamiltonian graph be planar and by modifying the method of determination of circuit C(k) with as many edges as possible, and from which the pseudo-Hamiltonian graph is defined. Using this improved algorithm, the rate of convergence of the algorithm is increased and the computer storage requirement is minimized. This algorithm also employs the edge T-matrix.

II. REVIEW OF EXISTING ALGORITHMS FOR TESTING THE PLANARITY OF A GRAPH

A. FISHER-WING ALGORITHM

1. Decomposition Theorem

Let C be a circuit in a graph G and G-C the subgraph of G that remains when the edges of C are deleted. (By the deletion of an edge we shall mean the removal of the edge but not the two vertices associated with the edge.) The edges of G-C are classified as follows:

1) Direct Connections: edges with both vertices in C
2) Edges of Attachment: edges with exactly one vertex in C
3) Exterior Edges: edges with no vertex in C.

We decompose G-C into union of edge disjoint subgraphs denoted as the "bridges" of C in G. The bridges may be defined by construction. To start, delete the vertices of C and all of the edges connected to these vertices. Next, group the subgraph of G that remains into a set of connected components. Denote each component which consists of a single isolated vertex by V_i, and each of the remaining components by G_i. Finally, associate with each component the set of edges of attachment which reconnect it to C. Note that this set may be null for some G_i, if G is not connected.

A bridge of C in G is then one of the following:

1) a direct connection to C (Type-1 bridge)
2) a set of edges of attachment which connect a vertex V_i to C (Type-2 bridge), or

3) a connected component, G_i, and the corresponding edges of attachment which connect G_i to C (Type-3 bridge).

Definition 1 A subgraph G' of G is called a pseudo-Hamiltonian graph if its decomposition with respect to a specified circuit C consists of bridges which are only Types 1 and 2.

Definition 2 A subgraph G_i'' of G is called a decomposed subgraph of G and is formed by the union of C and a bridge of Type-3.

Theorem 1 G is planar if and only if

(1) the pseudo-Hamiltonian graph G' is planar,
and
(2) each decomposed subgraph G_i'' is planar.

2. The Planarity of a Pseudo-Hamiltonian Graph and the Attachment Matrix

In Section 1 it is indicated that the problem of testing the planarity of an arbitrary graph could be reduced essentially to the problem of testing the planarity of a pseudo-Hamiltonian graph. A pseudo-Hamiltonian graph is planar if and only if its bridges can be mapped on the inside and outside of C in such a manner that no two edges on the same side cross. To determine whether such a mapping is possible the concept of alternation is introduced in the following paragraphs.

Definition 3 Let B be a bridge of C in G'. The vertices which are common to B and C are called the vertices of attachment of B.

Suppose B' is a bridge of C which is distinct from B, but possibly having the same vertices of attachment as B. B' does not alternate with B, if all of the vertices of attachment of B' lie on a path defined by two successive vertices of attachment of B.

Lemma 1 A pseudo-Hamiltonian graph G' is planar if and only if its bridges can be associated with two disjoint classes I and O, such that no two bridges in the same class alternate.

The property of alternation has a particularly straightforward representation in terms of the incidence matrix A of a pseudo-Hamiltonian graph G'. To determine whether two bridges alternate, it is needed only to examine the submatrix H of A of a pseudo-Hamiltonian graph introduced below.

Definition 4 The submatrix H of A of a pseudo-Hamiltonian graph G' whose rows correspond to the vertices of C, and whose columns correspond to the bridges is called the attachment matrix,

$$H = [A_1, A_2, \ldots, A_n],$$

where A_i is the submatrix of H which correspond to the bridge B_i in G'.

Theorem 2 Let A be the incidence matrix of a pseudo-Hamiltonian graph G'. Let H be the attachment matrix A. G' is planar if and only if H can be partitioned H = [I:O], where no two submatrices A_i, A_j in a partition I or O alternate.

3. Identification of a Planar Subgraph of an Arbitrary Graph

In this section the iterative algorithm described in Section 1 is utilized to extend the discussion to graphs which may be completely arbitrary. The decomposed subgraph which is tested in the k-th iteration is now denoted as G(k), where k=1 for G. To start the k-th iteration we find a circuit C(k), obtain the bridges of C(k) in G(k), and form the corresponding pseudo-Hamiltonian graph G'(k). If G'(k) is planar we proceed as in Section 1 and form the decomposed subgraphs for G(k). If G'(k) is nonplanar a set of non-planar edges N(k) is deleted so that G'(k)-N(k) is planar. The procedure is iterated until no decomposed subgraphs remain.

a. Formation of the Decomposed Subgraphs

If the edges of attachment have at least two distinct vertices in both C(k) and $G_i(k)$, the decomposed subgraph is denoted as the graph of Case 1 and is given by $C(k) \cup B_i(k)$, where $B_i(k)$ is the Type-3 bridge defined by $G_i(k)$. If the decomposed subgraph $C(k) \cup B_i(k)$ is separable or is not connected, it is noted as the graph of Case 2.

In the graph of Case 2 the planarity test needs only be applied to the component $B_i(k)$. This separable subgraph $B_i(k)$ will be denoted as $G(k+m)$, $m \geq 1$.

b. Determination of $C(k)$

Let $G(k)$ which is composed of $C(k-m)$ and $B_i(k-m)$ be the decomposed subgraph to be tested in the k-th iteration. As a preliminary step we first delete the edges of $G(k)$ whose vertices are of degree one. If there are parallel edges, these are deleted from $G(1)$ at the beginning of the algorithm.

Consider first the determination of $C(k)$ when $G(k)$ is the graph of Case 1. To construct $C(k)$ we first determine a pair of edges of attachment, (α,α') and (β,β') whose vertices α,β in $C(k-m)$ are successive and whose vertices α',β' in $G_i(k-m)$ are distinct. $C(k)$ is then given by

$$C(k) = (\alpha,\alpha') \cup P(\alpha',\beta') \cup (\beta',\beta) \cup P(\beta,\alpha)$$

where $P(\alpha',\beta')$ is a path from α' to β' which is contained entirely in $G_i(k-m)$, and $P(\beta,\alpha)$ is the path from β to α on $C(k-m)$. To construct $C(k)$ when $G(k)$ is the graph of Case 2 we select an edge (β',α') in $G_i(k-m)$ and find a long path $P(\alpha',\beta')$ in $G_i(k-m)-(\beta',\alpha')$. $C(k)$ is then given by

$$C(k) = (\beta',\alpha') \cup P(\alpha',\beta').$$

To insure that the edges of $C(k-m)$ are planar throughout the remaining iterations of the algorithm we specify that P' be placed first in the subsequent partitioning of the pseudo-Hamiltonian graph $G'(k)$.

4. Summary of the Iterative Algorithm in Terms of the Incidence Matrix

The incidence matrix for the k-th iteration is denoted by A(k) where k=1 for G. The algorithm identifies a planar subgraph of G, and is initialized by placing A(1) in the matrix list.

Step 1 Test if there is a matrix in the matrix list. If not, the run is over. If A(k) = A(1), delete the columns which correspond to parallel edges.

Step 2 Delete the rows and columns of A(k) which correspond to vertices of degree "0" and "1". If no columns remain, delete A(k) from the matrix list and return to Step 1.

Step 3 Find path $P(\alpha',\beta')$. If there is no path, place the connected components of $A(k) = A(k) - (\beta',\alpha')$ in the matrix list and return to Step 1.

Step 4 Form circuit C(k). If A(k) is associated with G(k) of Case 1, $C(k) = (\alpha,\alpha') \cup P(\alpha',\beta') \cup (\beta',\beta) \cup P(\beta,\alpha)$ and if A(k) is associated with G(k) of Case 2, $C(k) = (\beta',\alpha') \cup P(\alpha',\beta')$. Let A(k) be the matrix of $B_1(k)$. And rearrange the rows and columns of A(k) to correspond to the decomposition of the bridges of C(k).

Step 5 Partition the attachment matrix of A(k), placing edge P' first if A(k) is associated with G(k) of Case 1, If necessary, delete nonplanar edges.

Step 6 If the decomposition with respect to C(k) is pseudo-Hamiltonian, delete A(k) from the matrix list and return to Step 1. Otherwise, form the appropriate decomposed matrix for each Type-3 bridge of C(k). Place these matrices in the matrix list and delete A(k) from the matrix list. Return to Step 1.

B. LIN ALGORITHM

1. Generalization of Euler's Theorem

A fundamental theorem in solid geometry is Euler's theorem which relates the numbers of vertices, edges and faces of a polyhedron. Although Euler's theorem is customarily stated for polyhedrons, it is also applicable, with slight modification of definitions, to any planar graph.

Theorem 3 If G is a planar graph with v vertices ($v \geq 1$), e edges, P maximal connected subgraphs, and f faces, then

$$f = e - v + p + 1$$

Thoerem 4 For any planar graph G,

$$2e \geq \sum_{j=1}^{f} m_j$$

where m_j is the number of edges on the boundary of the j-th face.

2. Graphs With a Hamiltonian Circuit

Definition 5 A circuit containing all the vertices of a graph G is called a Hamilton circuit of G.

Given a graph containing a Hamilton circuit, we first redraw the graph with the Hamilton circuit as a convex polygon, and the remaining edges as line segments inside the polygon. By inspection of this graph, the edges inside the polygon can easily be divided into two sets:

(1) those with some crossings, designated by P, and

(2) those without any crossings, designated by Q.

Next, we try to sort the edges of P into two groups as follows:

(a) to begin with, both groups are empty. Arbitrarily add one edge of P to Group 1.

(b) add to Group 2 all the edges of P which cross any of the edges in Group 1 so far.

(c) add to Group 1 all the edges of P which cross any of the edges in Group 2 so far.

(d) repeat (2) and (3) until all the edges of P are used.

If the above process exhausts all the edges of P without any one appearing in both Group 1 and Group 2, then the given graph is planar. We can in this case draw the edges of Group 1 inside the polygon, those of Group 2 outside the polygon, and the edges of Q either inside or outside the polygon. On the other hand, if at any step an edge appears in both Group 1 and Group 2, then the given graph is nonplanar.

3. The General Case

From the specified incidence relationship among the edges and vertices of a graph G, G can be drawn on a plane. Assume that there are some crossings of edges at points other than the vertices. For the question at hand, the following preparation steps can be justified easily:

(a) Any series connection or parallel connection of edges is replaced by a single edge.

(b) If the graph is separable, decompose the graph into its components. The given problem may be treated as several problems, each of which is concerned with one component of the graph.

(c) If the non-separable graph contains a connected subgraph G_1 which has exactly two vertices (i,j) in common with the complement of G_1, then we shall first investigate G_1. If G_1 is nonplanar after adding an edge between (i,j), then G is nonplanar. If G_1 is a planar one-terminal-pair graph with respect to (i,j), then G_1 may be replaced by a single edge between the vertices (i,j), and the investigation continues.

After the above preparation steps it is assumed that the graph G under investigation is non-separable, contains no series or parallel connection of edges, and contains no subgraph which has only two vertices in common with its complement, unless the latter is a single edge. The method of determining whether G is planar now proceeds step by step as follows:

(1) Select a circuit C in G. If C contains all the vertices of G, then C is a Hamilton circuit of G, and the method of Section 2 may be used.

(2) Assume that C does not contain all of the vertices of G. Let Q be the set of all edges of G which do not belong to C, and have both endpoints on C. C + Q (the union of C and Q) may be tested by the method of Section 2. If C + Q is nonplanar, then G is nonplanar.

(3) Assume that C + Q is planar. Let P be the set of all edges of G with exactly one endpoint on C. Let G_s be the subgraph obtained from G by the removal of P, excluding the endpoints of P, and the removal of C + Q, including the endpoints of C + Q. G_s may contain some isolated vertices. Decompose G_s into its maximal connected subgraphs H_1, H_2, H_3, etc.

(4) Assume that G_s contains at most two maximal connected subgraphs H_1 and H_2. We separate P into $P = P_1 + P_2$, such that each edge of P_1 and P_2 has one endpoint on C and the other endpoint on H_1 and H_2 respectively. Determine whether $C + P_1 + H_1$ and $C + P_2 + H_2$ are planar respectively. If either $C + P_1 + H_1$ or $C + P_2 + H_2$ is nonplanar, then G is nonplanar.

(5) Assume that both $C + P_1 + H_1$ and $C + P_2 + H_2$ are planar. Then $C + P_1 + H_1 + P_2 + H_2$ is planar. The planar graph $C + P_1 + H_1 + P_2 + H_2$ divides the plane into f faces. If every edge of Q joins two vertices on the boundary of the same face, then G is planar. Otherwise G is nonplanar.

C. PHUNG-CHAN ALGORITHM

1. The Edge T-Matrix and Its Properties

The edge T-matrix is introduced by Phung and Chan [10] and the Fisher-Wing algorithm was improved using the edge T-matrix. Its definition and some of its properties will be repeated in this section to facilitate the discussion of the algorithm that follows.

Definition 6 Given a graph G of v vertices. The edge T-matrix, denoted by $\underline{T}$, associated with G is a triangular array of v-1 rows and columns, of which the i-th row corresponds to vertex V_{i+1}, and the j-th column to vertex V_j. The entry t_{ij} in the i-th row and j-th column is:

$$t_{ij} = \begin{cases} \text{sum of edge designations between } V_{i+1} \text{ and } V_j, \\ \text{"0" if } V_{i+1} \text{ and } V_j \text{ are disconnected.} \end{cases}$$

Property 1

(a) The number N of nonzero entries in row i-1 and column i is the order of vertex V_i. In particular if N=0, then V_i is an isolated vertex.

(b) The set of nonzero entries, denoted by R_{i-1} for $i \geq 2$, in the i-1 row corresponds to the set of edges connecting V_i to V_j, for $1 \leq j \leq i-1$: the set of nonzero entries C_i in the i column corresponds to the set of edges connecting V_i to V_j with $i+1 \leq j \leq v$.

Property 2

(a) If all entries t_{ij} in $\lfloor T$ are zero for $k \leq i \leq v-1$ and $1 \leq j \leq k$, then the graph associated with $\lfloor T$ is disconnected.

(b) If all of these entries are zero except one, then the graph corresponding to $\lfloor T$ is composed of two separable subgraphs.

Property 3 A set of v-1 nonzero entries taken one from each row of the edge T-matrix $\lfloor T$, associated with a graph G, at a time is a complete set of branches of a tree of G.

Definition 7 Given a graph G. An Euler tree of G, when it exists, is a particular tree which is formed by a path visiting all vertices of G.

Property 4 There exists an Euler tree in a graph G if in the edge T-matrix associated with G, the set of entries t_{ii} are all nonzero $1 \leq i \leq v-1$.

Definition 8 Consider a graph of vertices. The edge T-matrix associated with the graph, denoted by $\lfloor Te$, is called the effective edge T-matrix if

(1) $t_{ii} \neq 0$ for $1 \leq i \leq k$

(2) $t_{ij} = 0$ for $k+1 \leq i \leq v-1$, $k+1 \leq j \leq v-1$

Definition 9 Consider a graph G of v vertices. Suppose that the edge T-matrix $\lfloor T$ associated with G is an effective edge T-matrix. The tree of G whose branches are represented by

(1) t_{ii} for $1 \leq i \leq k$

(2) $t_{pq'}$ for $k+1 \leq p \leq v-1$

is called a pseudo-Euler tree, where $t_{pq'}$ is the last nonzero entry in row p.

The k branches represented by t_{ii} form a path connecting V_1 to V_{k+1}. This path is called the trunk of the tree, and the remaining branches associated with $t_{pq'}$ are known as main branches of the tree.

Definition 10 Given the edge T-matrix $T_1 = [t_{pq}]$ associated with a graph G. It is said that the edge T-matrix $T_{1j} = [t'_{pq}]$ is derived from T_1 by applying the C-transformation using column $j \geq 2$ as operating column, if:

(1) $t'_{pq} = t_{pq}$ for $1 \leq p,\ q \leq j-2$

(2) $t'_{pq} = t_{p+1,q}$ for $j-1 \leq p \leq v-2$ and $1 \leq q \leq j-1$

(3) $t'_{pq} = t_{p+1,q+1}$ for $j \leq p \leq v-2$ and $j \leq q \leq v-2$

$$(4)\quad t'_{v-1,q} = \begin{cases} t_{j-1,q} & \text{for } 1 \leq q \leq j-1 \\ t_{qj} & \text{for } j \leq q \leq v-1 \end{cases}$$

In particular for j=1, then:

(5) $t'_{pq} = t_{p+1,q+1}$ for $1 \leq p,\ q \leq v-2$

(6) $t'_{v-1,q} = t_{q1}$ for $1 \leq q \leq v-1$

Definition 11 A column k in the edge T-matrix is called a C*-operating column if:

(1) $t_{k-1,k-1} = 0$ and

(2) $\sum_{i=k}^{v-1} t_{i,k-1} \neq 0$

The C-transform in which the operating column is a C*-operating column is called the C*-transform.

Definition 12 An edge T-matrix is called a resolving edge T-matrix if no column in this edge T-matrix can be used as a C*-operating column.

Lemma 2 Any edge T-matrix can be transformed into a resolving edge T-matrix.

2. Planarity of a Pseudo-Hamiltonian Graph From its Edge T-Matrix

Lemma 3 A pseudo-Hamiltonian graph is planar if and only if the set of chords defined with respect to the pseudo-Euler tree can be mapped on the plane of the tree without cross over.

Definition 13 Two chords with respect to apseudo-Euler tree is said to be of

(1) Class-1 chord if it is a chord with both vertices in the trunk, or

(2) Class-2 chord if it is a chord with only one vertex in the trunk.

Lemma 4 Two Class-1 chords with respect to a pseudo-Euler tree when mapped on the same plane with the tree, alternate if their vertices alternate on the trunk of the tree.

Lemma 5 Let V_j and V_k be respectively the attachment vertex of the main branches which connect the C-isolated vertices V_p and V_q to the trunk. Let e_{pr} and e_{qs} be two Class-2 chords connecting respectively V_p and V_q to the trunk. Then e_{pr} and e_{qs} alternate when mapped on the same plane with the tree if vertices V_j and V_r alternate with V_k and V_s on the trunk of the pseudo-Euler tree.

Lemma 6 Let V_j be the attachment vertex of the main branch which connects a C-isolated vertex V_q. Consider a Class-1 chord e_{rs} and Class-2 chord e_{qp}. Then e_{rs} and e_{qp} alternate if V_r and V_s alternate with V_j and V_p on the trunk.

Thoerem 5 A pseudo-Hamiltonian graph is planar if and only if the set of chords defined with respect to a pseudo-Euler tree of the graph can be partitioned into two subsets such that no two chords in the same subset alternate.

3. Determination of a Planar Subgraph of an Arbitrary Graph

Let the decomposed subgraph, which is being tested, in the k-th iteration be G(k), where k=1 for the given graph. To start the k-th iteration, a circuit C(k) is first to be found, then from which a corresponding pseudo-Hamiltonian graph G'(k) and a set of decomposed subgraphs, denoted by G(k+m), for m=1,2,..., are also to be obtained. If G'(k) is nonplanar, a set of nonplanar edges N(k) will be formed. The procedure is iterated until no decomposed subgraphs remain.

a. Determination of C(k)

Suppose that the graph is associated with a resolving edge T-matrix $\lfloor T_r$. The resolving tree from $\lfloor T_r$ is associated with the branches that are:

(1) All nonzero entries t_{ii}, and

(2) all nonzero entries $t_{pq'}$ if $t_{ii}=0$, and $t_{pq'}$ is the last nonzero entry in row p.

The remaining nonzero entries of $\lfloor T_r$ represent chords with respect to the resolving tree T_r.

Definition 15 With respect to a resolving tree, chords are divided into:

(1) Class-1 chord, if it has both vertices in the same path,

(2) Class-2 chord, if it has only one vertex in a path and the other is reconnected to the same path by a tree branch,

(3) Class-3 chord, if it connects two connected components.

Rule 1 The fundamental circuit obtained from Class-1 chord t_{pq} is:

$$C = (t_{qq}, t_{q+1,q+1}, \ldots, t_{pp}, t_{pq})$$

Rule 2 The fundamental circuit obtained from a Class-2 chord t_{pq} is:

$$C = (t_{qq}, t_{q+1,q+1}, \ldots, t_{q'-1,q'-1}, t_{pq'}, t_{pq})$$

Rule 3 Let $P_{i,i+r}$ and $P_{j,j+s}$ be respectively two paths which are connected together by a tree branch represented by $t_{j-1,q'}$. Let t_{pq} be a Class-3 chord connecting these paths. Then the fundamental circuit obtained from t_{pq} is:

$$C = (t_{qq}, t_{q+1,q+1}, \ldots, t_{q'-1,q'-1}, t_{j-1,q'}, t_{jj}, t_{j+1,j+1}, \ldots$$
$$\ldots, t_{pp}, t_{pq})$$

where $i < q < q' < i+r$ and $j < p < j+s$, or

$$C = (t_{j-1,q'}, t_{q'q'}, t_{q'+1,q'+1}, \ldots, t_{q-1,q-1}, t_{pq}, t_{pp},$$
$$t_{p-1,p-1}, t_{p-2,p-2}, \ldots, t_{jj})$$

where $i < q' < q < i+r$ and $j < p < j+s$

Remark 1 In finding the fundamental circuits of a Class-3 chord connecting path $P_{i,i+r}$ and $P_{j,j+s}$, it is observed that there is no tree branch connecting $P_{i,i+r}$ and $P_{j,j+s}$. In this case a train of tree branches which has common node with both $P_{i,i+r}$ and $P_{j,j+s}$ is required. In general such a train of tree branches can be obtained, since the resolving tree is by definition a connected graph. Phung and Chan have given an algorithm for finding the fundamental circuits with respect to a resolving tree of general case [8].

Using Rules 1, 2 and 3, fundamental circuits with respect to a resolving tree of G(k) can be obtained and the circuit C(k) of G(k) is properly selected among fundamental circuits according to whether G(k) is separable or non-separable.

b. Determination of the Edge T-Matrix Associated with G'(k)

Let G'(k) be the pseudo-Hamiltonian graph which results from the decomposition of G(k) with respect to the specified circuit C(k). The edge T-matrix associated with G'(k) can be determined as follows.

(1) Use C-transform to transform $\underline{|T_r(k)}$ into another resolving edge T-matrix $\underline{|T_r(k)}$ of which the first entries t_{ii} for $1 \leq i \leq s$ are associated with the ordered sequences of edges of C(k) and where s is the number of edges of C(k).

(2) Identify isolated vertices V_i and vertices of each of the connected components which result from the decomposition of G(k) with respect to C(k). Add all entries in the same column of the first s columns and rows corresponding to vertices of the same connected component $G_i(k)$ to form a new row.

(3) The effective edge T-matrix $\underline{|T_e'(k)}$ associated with G'(k) is formed by the first s rows associated with C(k) and rows associated with C-isolated vertices and rows which correspond individually to a new vertex obtained by identifying vertices of a connected component.

c. Determination of the Edge T-Matrix Associated with a Decomposed Subgraph $G(k+m)$

The edge T-matrix associated with a decomposed subgraph $G_i(k)$ can be obtained directly from $\lfloor T_r(k)$ by letting all of the entries corresponding to bridges of $C(k)$ in $\lfloor T_r(k)$ equal to zero except for the entries corresponding to the Type-3 bridge $B_i(k)$ from which $G_i(k)$ is defined and removing all rows and columns associated with the isolated vertices and the vertices in connected components except for $G_i(k)$ from this reduced matrix.

4. Algorithm for Identifying a Planar Subgraph From an Arbitrary Graph

The algorithm is initialized by placing $\lfloor T(1)$ of $G(1)$ in the matrix list. At the beginning of the algorithm nonplanar edges $N(1) = 0$.

Step 1 Test if there is an edge T-matrix in the matrix list. If not, the run is over.

Step 2 Transform $\lfloor T(k)$ into $\lfloor T_r(k)$.

Step 3 If $G(k)$ is the graph of Case 1, select a circuit $C(k)$ using the process given in Section 3. If $G(k)$ is the graph of Case 2, return to step 2 with $\lfloor T(k) = \lfloor T_{B_i}(k-m)$ where $B_i(k-m)$ denotes the Type-3 bridge from which $G_i(k-m)$ is defined. If no circuit can be formed, delete $\lfloor T(k)$ from the matrix list and then return to Step 1.

Step 4 Implement the edge T-matrix associated with $G'(k)$ and test the planarity of $G'(k)$. Place nonplanar edge in $N(k)$.

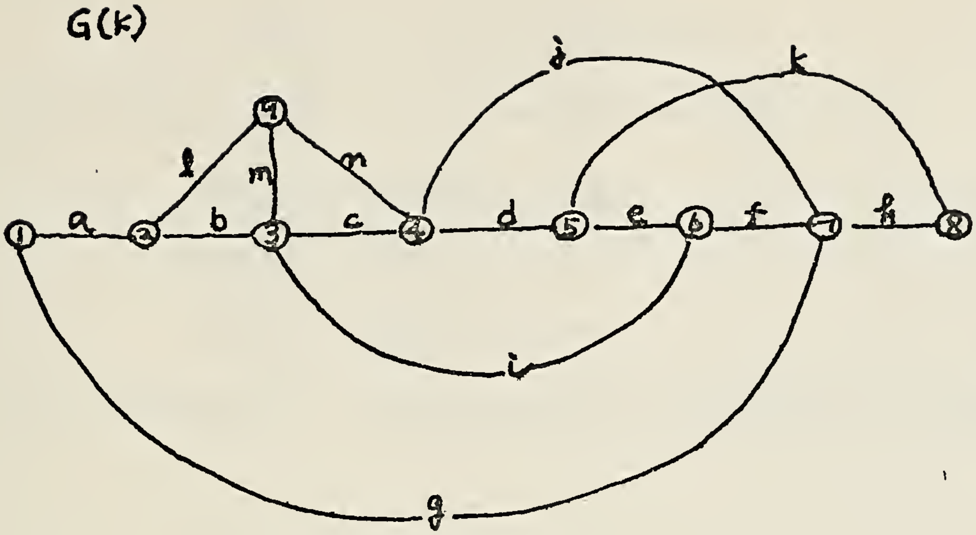
Step 5 If the decomposition of G(k) with respect to C(k) is a pseudo-Hamiltonian, delete $\underline{\vert T(k)}$ from the matrix list and then return to Step 1. Otherwise, implement an edge T-matrix associated with each Type-3 bridge of C(k). Place these edge T-matrices in the matrix list and then delete $\underline{\vert T(k)}$ from the matrix list. Return to Step 1.

III. MODIFICATION OF PHUNG-CHAN ALGORITHM

Planarity test and planar partition of a graph G(k) given in the following examples will be done by using the Phung-Chan method. Doing the examples given below, it will be pointed out that the application of Theorem 5 which is established by Phung and Chan depends on the specified pseudo-Euler tree and that the Phung-Chan algorithm does not converge in some cases. The algorithm for determining a proper pseudo-Euler tree of G'(k) will be given to make Theorem 5 work and also the algorithm for finding C(k) of G(k) will be given to make the algorithm converge in any case of edge T-matrices associated with G(k). Finally, modification of Phung-Chan Algorithm will be given.

A. ILLUSTRATION OF PHUNG-CHAN ALGORITHM

Example 1 Planarity test and planar partition of the graph G(k) given below will be done by using Phung-Chan method. Numbers in small circles in G(k) represent the vertices of G(k). The G(k) can be assumed the pseudo-Hamiltonian graph obtained by decomposition of an arbitrary graph with respect to the circuit whose edges are a,b,c,d,e,f and g and so vertex ⑨ in G(k) is assumed to identify a connected component $G_i(k-m)$ or a C-isolated vertex V_i.



Step 1 It is assumed that $\lfloor T(k)$ is selected from the matrix list.

$$\lfloor T(k) = \begin{array}{c|cccccccc} 2 & a & & & & & & & \\ 3 & . & b & & & & & & \\ 4 & . & . & c & & & & & \\ 5 & . & . & . & d & & & & \\ 6 & . & . & i & . & e & & & \\ 7 & g & . & . & j & . & f & & \\ 8 & . & . & . & . & k & . & h & \\ 9 & . & l & m & n & . & . & . & . \\ \hline & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} = \lfloor T_r(k)$$

Step 2 Completed in Step 1

$T_r(k)$

Step 3 The branches of the resolving tree $T_r(k)$ of are:

$$T_r(k) = (a,b,c,d,e,f,h,n)$$

The chords defined with respect to $T_r(k)$ are:

Class-1 chords: g,i,j,k

Class-2 chords: l,m

Class-3 chords: none

The fundamental circuits obtained according to chords are

$C_1 = (\underline{g},f,e,d,c,b,a)$

$C_2 = (\underline{i},e,d,c)$

$C_3 = (\underline{j},f,e,d)$

$C_4 = (\underline{k},h,f,e)$

$C_5 = (\underline{l},n,c,b)$

$C_6 = (\underline{m},n,c)$

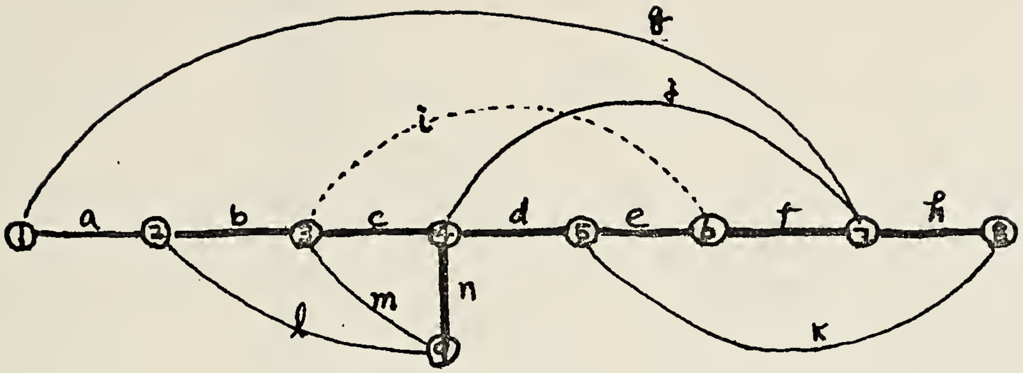
From the above fundamental circuits it is found that C_1 is the one which has the greatest number of edges among them and so C_1 is selected as circuit $C(k)$.

Step 4 The planarity test of $G'(k)$ $(=G(k))$ yields:

$L(k) = (k,l,m)$

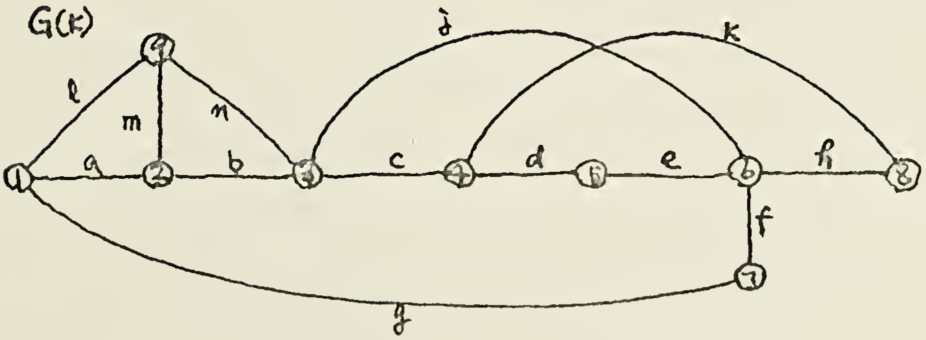
$U(k) = (g,j)$

$N(k) = (i)$



The graph G(k) is nonplanar because the chord i cannot be mapped on the same plane with respect to the chosen pseudo-Euler tree without cross over. The bold line in the above graph is the chosen pseudo-Euler tree. According to Step 4 test the planar subgraph of G(k) is composed of the edge "i". The nonplanar edge i is represented as dotted line in the above graph.

Example 2 Planarity test and planar partition of the graph G(k) given below will be done by using the Phung-Chan algorithm.



Step 1 It is assumed $\underline{|T(k)}$ associated with G(k) is selected from the matrix list.

$$\underline{|T(k)} = \begin{array}{c|cccccccc}
2 & a \\
3 & . & b \\
4 & . & . & c \\
5 & . & i & . & d \\
6 & . & . & j & . & e \\
7 & g & . & . & . & . & f \\
8 & . & . & . & k & . & h & . \\
9 & l & m & n & . & . & . & . & . \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array} = \underline{|T_r(k)}$$

Step 2 Completed in Step 1.

Step 3 The branches of the resolving tree $T_r(k)$ of $\underline{|T_r(k)}$ are:

$T_r(k) = (a,b,c,d,e,f,h,n)$

The chords defined with respect to the above resolving tree $T_r(k)$ are:

Class-1 chords: i,j,g

Class-2 chords: l,m,k

The fundamental circuits obtained according to the chords are:

$C_1 = \underline{i},d,c,b$

$C_2 = \underline{j},e,d,c$

$C_3 = \underline{g},f,e,d,c,b,a$

$C_4 = \underline{l},n,b,a$

$C_5 = \underline{m},n,b$

$C_6 = \underline{k},h,e,d$

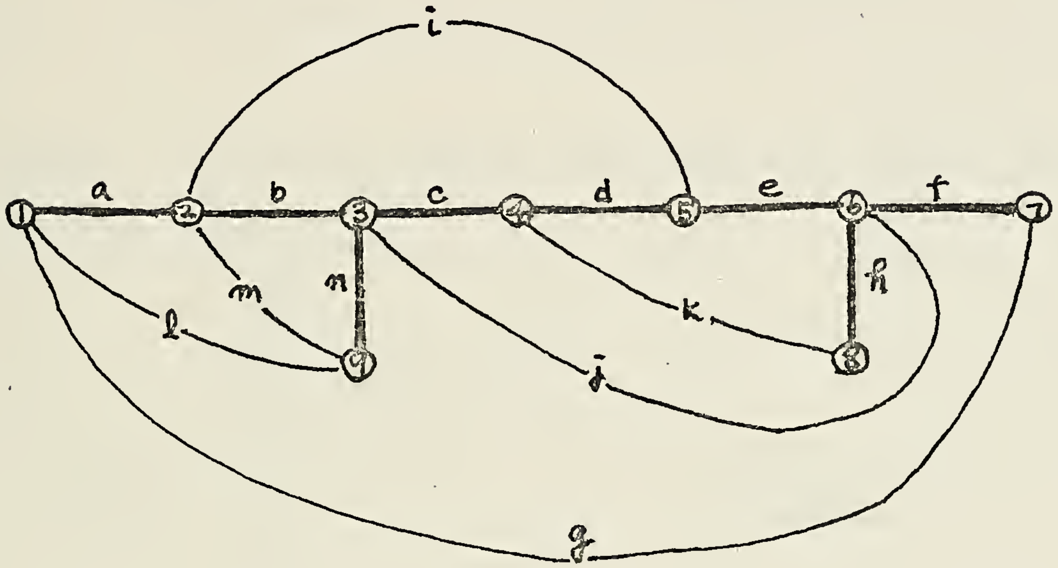
From the above fundamental circuit list C_3 is selected as circuit C(k).

Step 4 The planarity test of G'(k) (=G(k)) yields:

L(k) = (l,m,k,g,j)

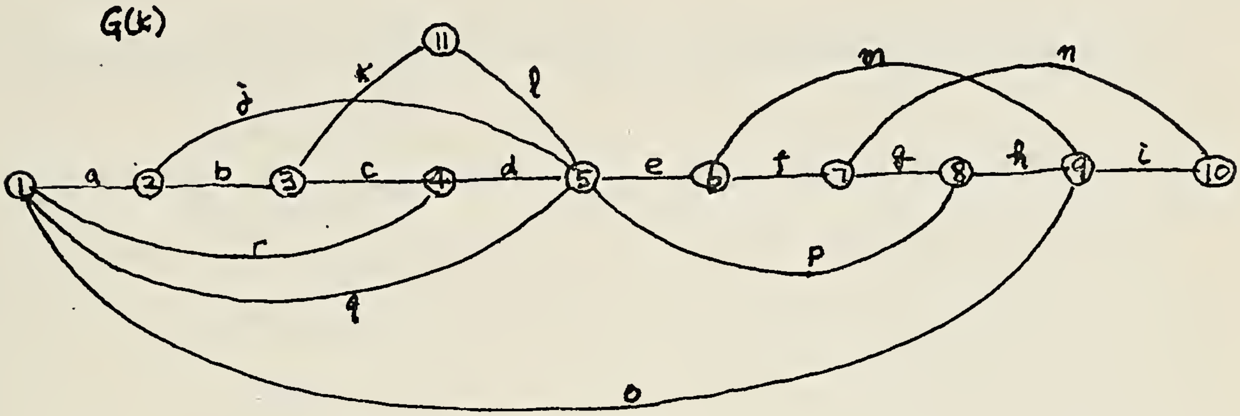
U(k) = (i)

N(k) = none



The chords with respect to the chosen resolving tree can be mapped on the same plane without cross over. The graph G(k) is planar. The bold line in the above graph is the chosen pseudo-Euler tree.

Example 3 Planarity test and planar partition of the graph G(k) given below will be done by using the Phung-Chan method.



Step 1 It is assumed $\lfloor\underline{T(k)}$ associated with G(k) is selected from the matrix list.

$$\lfloor\underline{T(k)} = \begin{array}{r|cccccccccc}
2 & a \\
3 & . & b \\
4 & r & . & c \\
5 & q & j & . & d \\
6 & . & . & . & . & e \\
7 & . & . & . & . & . & f \\
8 & . & . & . & . & p & . & g \\
9 & o & . & . & . & . & m & . & h \\
10 & . & . & . & . & . & . & n & . & i \\
11 & . & . & k & . & l & . & . & . & . & . \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array} = \lfloor\underline{T_r(k)}$$

Step 2 Completed in Step 1

Step 3 The branches of the resolving tree $T_r(k)$ of $\lfloor\underline{T_r(k)}$ are:

$T_r(k) = (a,b,c,d,e,f,g,h,i,l)$

The chords defined with respect to $T_r(k)$ are:

Class-1 chords: j,m,n,o,p,q,r

Class-2 chords: k

The fundamental circuits obtained according to chords are:

$C_1 = \underline{j},d,c,b$

$C_2 = \underline{m},h,g,f$

$C_3 = \underline{n},i,h,g$

$C_4 = \underline{o},h,g,f,e,d,c,b,a$

$C_5 = \underline{p},q,f,e$

$C_6 = \underline{q},d,c,b,a$

$C_7 = \underline{r},c,b,a$

$C_8 = \underline{k},l,d,c$

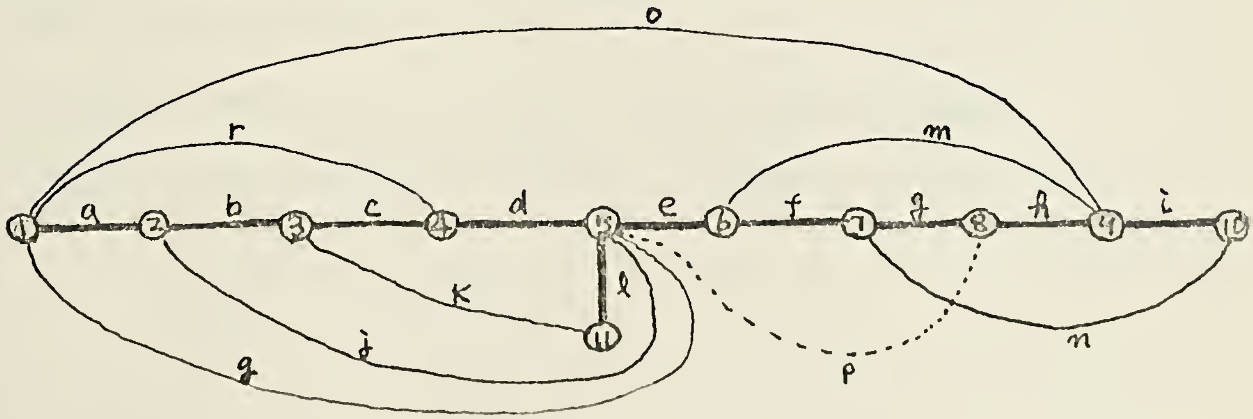
From the fundamental circuits list it is found that C_4 is the one which has the greatest number of edges among them and C_4 is selected as circuit C(k).

Step 4 The planarity test of G'(k) (=G(k)) yields:

L(k) = (k,j,q,n)

U(k) = (r,o,m)

N(k) = (p)



Because of nonplanar edge p the graph G(k) is nonplanar. According to the above test the planar subgraph of G(k) is composed of the edge "p". The bold line in the above graph is the chosen pseudo-Euler tree. Nonplanar edge p is represented as dotted line in the above graph.

Example 4 The planarity of the graph associated with the edge T-matrix given below is tested by using the Phung-Chan method.

$$\underline{|T(k)} = \begin{array}{r|cccccccccc}
3 & b & & & & & & & & & \\
4 & . & c & & & & & & & & \\
5 & j & . & d & & & & & & & \\
6 & . & . & . & e & & & & & & \\
7 & . & . & . & . & f & & & & & \\
8 & . & . & . & p & . & g & & & & \\
9 & . & . & . & . & m & . & h & & & \\
1 & a & . & r & q & . & . & . & o & & \\
10 & . & . & . & . & . & n & . & i & . & \\
11 & . & k & . & l & . & . & . & . & . & . \\
\hline
 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 & 10
\end{array} = \underline{|T_r(k)}$$

Step 1 It is assumed that $\underline{|T(k)}$ associated with G(k) is selected from the matrix list.

Step 2 Completed in the given edge T-matrix $\underline{|T(k)}$.

Step 3 The branches of the resolving tree $T_r(k)$ are:

$$T_r(k) = (b,c,d,e,f,g,h,o,i,l)$$

The chords defined with respect to the above resolving tree $T_r(k)$ are:

Class-1 chords: j,p,m,a,r,q

Class-2 chords: n,k

The fundamental circuits obtained according to the chords are:

$C_1 = \underline{j},d,c,b$

$C_2 = \underline{p},g,f,e$

$C_3 = \underline{m},h,g,f$

$C_4 = \underline{a},o,h,g,f,e,d,c,b$

$C_5 = \underline{r},o,h,g,f,e,d$

$C_6 = \underline{q},o,h,g,f,e$

$C_7 = \underline{n},i,h,q$

$C_8 = \underline{k},l,d,c$

From the above fundamental circuit list C_4 is selected as C(k)

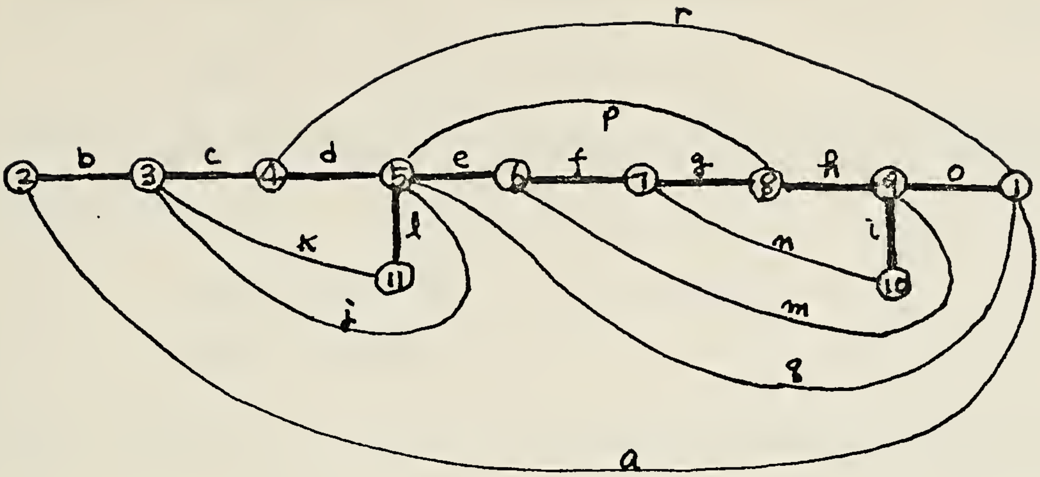
Step 4 The planarity test of $\underline{|T'(k)}$ (= $\underline{|T(k)}$) yields:

L(k) = (k,n,a,q,m,j)

U(k) = (r,p)

N(k) = none

According to Step 4 test the graph associated with the given edge T-matrix $\underline{|T(k)}$ is planar. The bold line in the following graph is the chosen pseudo-Euler tree.



Example 5 The planarity of the graph associated with edge T-matrix given below is tested by using the Phung-Chan method.

$$\underline{|T(k)} = \begin{array}{r|llllllllll}
7 & f \\
8 & . & g \\
9 & m & . & h \\
1 & . & . & . & o \\
2 & . & . & . & . & a \\
3 & . & . & . & . & . & b \\
4 & . & . & . & . & r & . & c \\
5 & e & . & p & . & q & j & . & d \\
11 & . & . & . & . & . & . & k & . & l \\
10 & . & n & . & i & . & . & . & . & . & . \\
\hline
 & 6 & 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 11
\end{array} = \underline{|T_r(k)}$$

Step 1 It is assumed that the given edge T-matrix $\underline{|T(k)}$ is selected from the matrix list.

Step 2 Completed in the given edge T-matrix $\underline{|T(k)}$.

Step 3 The branches of the resolving tree $T_r(k)$ of $\underline{|T_r(k)}$ are:

$T_r(k) = (f,g,h,o,a,b,c,d,l,i)$

The chords defined with respect to the above resolving tree $T_r(k)$ are:

Class-1 chords: m,r,j,k,e,p,q

Class-2 chords: n

The fundamental circuits obtained according to the chords are:

$C_1 = \underline{m},h,g,f$

$C_2 = \underline{r},c,b,a$

$C_3 = \underline{j},d,c,b$

$C_4 = \underline{k},l,d,c$

$C_5 = \underline{p},d,c,b,a,o,h$

$C_6 = \underline{e},d,c,b,a,o,h,g,f$

$C_7 = \underline{q},d,c,b,a$

$C_8 = \underline{n},i,h,g$

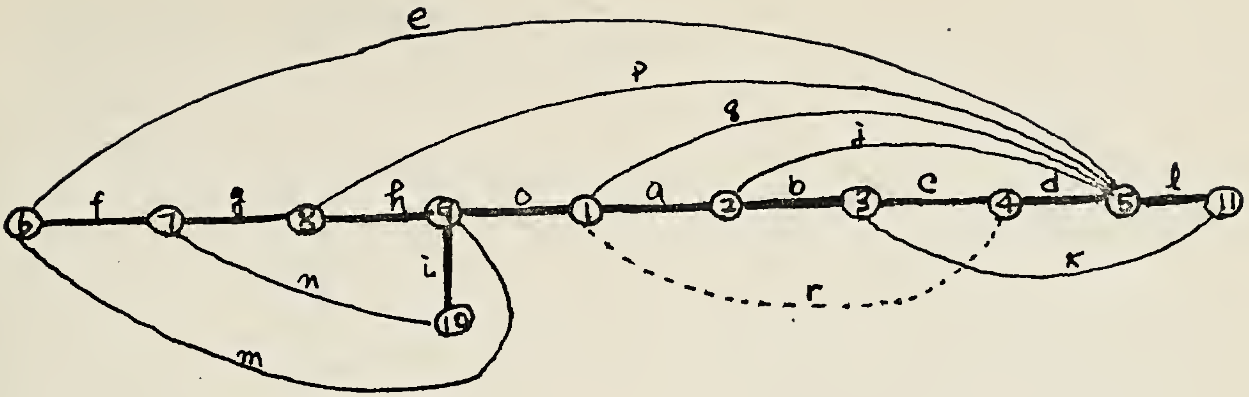
From the above fundamental circuit list C_6 is selected as C(k).

Step 4 The planarity test of $\underline{|T'(k)}$ (= $\underline{|T(k)}$) yields:

L(k) = n,k,m

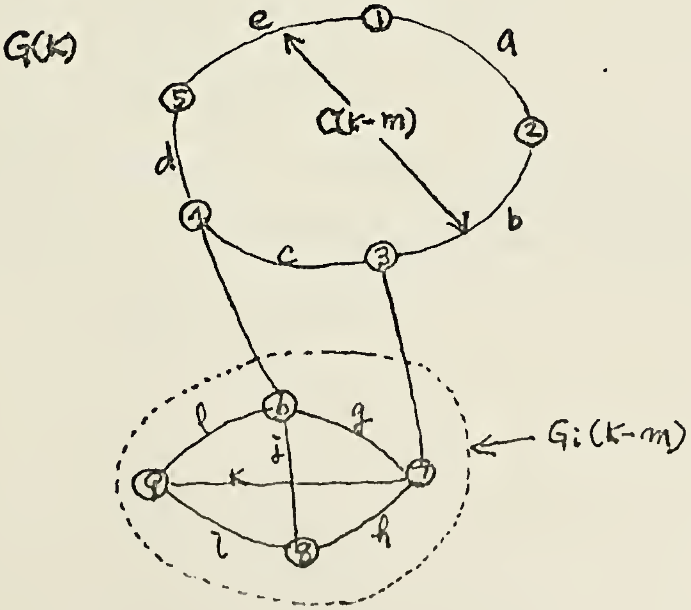
U(k) = e,p,q,j

N(k) = r



According to tep 4 test, the graph associated with the given edge T-matrix is nonplanar. The bold line in the above graph is the chosen pseudo-Euler tree. Nonplanar edge r is represented as a dotted line in the above graph.

Example 6 The graph G(k) given below is composed of circuit C(k-m) and a Type-3 bridge B_i(k-m). The G(k) is assumed the decomposed subgraph to be tested in the kth iteration. Planarity test of G(k) is done by using the Phung-Chan method.



For the iteration

Step 1

$$\underline{|T(k)} = \begin{array}{c|cccccccc} 2 & a \\ 3 & . & b \\ 4 & . & . & c \\ 5 & e & . & . & d \\ 6 & . & . & . & f & . \\ 7 & . & . & m & . & . & g \\ 8 & . & . & . & . & . & j & h \\ 9 & . & . & . & . & . & l & k & i \\ \hline & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} = \underline{|T_r(k)}$$

Step 2 Completed in Step 1.

Step 3 The branches of the resolving tree $T_r(k)$ obtained from $\underline{|T_r(k)}$ are:

$$T_r(k) = (a,b,c,d,f,g,h,i)$$

The chords specified with respect to $T_r(k)$ are:

Class-1 chords: e,j,k,l

Class-2 chords: none

Class-3 chords: m

In this case there is only one class-3 chord "m" and so the C(k) is the fundamental circuit specified by chord "m".

$$C(k) = \underline{m}\ g\ f\ c$$

Step 4 Implement the edge T-matrix associated with G'(k) and test the planarity of G'(k). G'(k) is the subgraph obtained from G(k) by identifying the vertices in each of the connected components $G_i(k)$ with respect to the specified circuit C(k).

$$\underline{\lvert T(k)} = \begin{array}{c|cccccccc}
4 & c \\
6 & . & f \\
7 & m & . & g \\
1 & . & . & . & . \\
2 & b & . & . & . & a \\
5 & . & d & . & . & e & . \\
8 & . & . & j & h & . & . & . \\
9 & . & . & l & k & . & . & . & i \\
\hline
 & 3 & 4 & 6 & 7 & 1 & 2 & 5 & 8
\end{array}
\qquad
\underline{\lvert T_r(k)} = \begin{array}{c|cccccccc}
4 & c \\
6 & . & f \\
7 & m & . & g \\
8 & . & . & j & h \\
9 & . & . & l & k & i \\
1 & . & . & . & . & . & . \\
2 & b & . & . & . & . & . & a \\
5 & . & d & . & . & . & . & e & . \\
\hline
 & 3 & 4 & 6 & 7 & 8 & 9 & 1 & 2
\end{array}$$

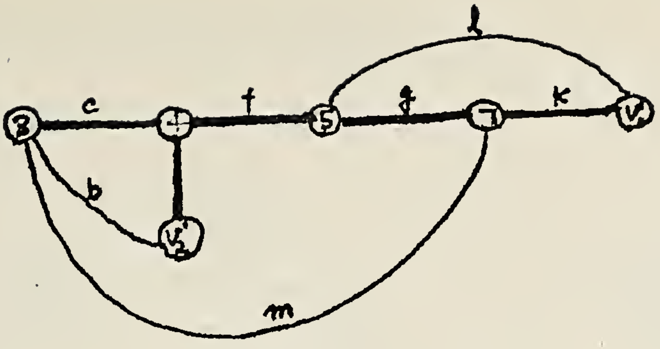
Note: $\underline{\lvert T(k)}$ here is another edge T-matrix of G(k). Vertices of $\underline{\lvert T(k)}$ in Step 1 are rearranged according to specified circuit C(k).

$$\underline{\lvert T'_r(k)} = \begin{array}{c|ccccc}
4 & c \\
6 & . & f \\
7 & m & . & g \\
V'_1 & . & . & l & k \\
V'_2 & b & d & . & . & . \\
\hline
 & 3 & 4 & 6 & 7 & V'_1
\end{array}$$

L(k) = b,m

U(k) = l

N(k) = none



where V_i' is the vertex obtained by shrinking the connected component $G_i(k)$ to a point. The edge j parallel to the edge ℓ and the edge h parallel to the edge k are deleted from the $\lfloor T_r'(k)$. The pseudo-Euler tree of $\lfloor T_r'(k)$ is drawn with bold line.

According to the above test the decomposed subgraph G'(k) is planar.

Step 5 Implement edge T-matrices associated with the decomposed subgraph $G_i''(k)$ which is formed with C(k) and Type-3 Bridge $B_i(k)$. Place these edge T-matrices in the matrix list. Delete $\lfloor T(k)$ from the matrix list and return to Step 1.

$$\lfloor T(k+1) = \begin{array}{c|ccccc} 4 & c & & & & \\ 6 & . & f & & & \\ 7 & m & . & g & & \\ 8 & . & . & j & h & \\ 9 & . & . & l & k & i \\ \hline & 3 & 4 & 6 & 7 & 8 \end{array} \qquad \lfloor T(k+2) = \begin{array}{c|cccccc} 4 & c & & & & & \\ 6 & . & f & & & & \\ 7 & f & . & g & & & \\ 1 & . & . & . & . & & \\ 2 & b & . & . & . & a & \\ 5 & . & d & . & . & e & . \\ \hline & 3 & 4 & 6 & 7 & 1 & 2 \end{array}$$

For the k+1-th iteration

Step 1

$$\lfloor\underline{T(k+1)} = \begin{array}{r|lllll} 4 & c & & & & \\ 6 & . & f & & & \\ 7 & m & . & g & & \\ 8 & . & . & j & h & \\ 9 & . & . & l & k & i \\ \hline & 3 & 4 & 6 & 7 & 8 \end{array} = \lfloor\underline{T_r(k+1)}$$

Step 2 Completed in Step 1.

Step 3 The branches of the resolving tree $T_r(k+1)$ obtained from $\lfloor\underline{T_r(k+1)}$ are:

$$T_r(k+1) = (c,f,g,h,i)$$

The chords specified with respect to $T_t(k+1)$ are:

Class-1 chords: m,j,k,l

Class-2 chords: none

Class-3 chords: none

The fundamental circuits obtained according to the chords with respect to the specified resolving tree $T_r(k+1)$ are:

$C_1 = \underline{m},g,f,c$

$C_2 = \underline{l},i,h,g$

$C_3 = \underline{k},i,h$

$C_4 = \underline{j},h,g$

From the above fundamental circuits C_2 is selected as circuit C(k+1).

Step 4 Implement the edge T-matrix associated with G'(k+1) and test the planarity of G'(k+1).

$$\underline{|T(k+1)} = \begin{array}{c|ccccc} 7 & g & & & & \\ 8 & j & h & & & \\ 9 & l & k & i & & \\ 3 & . & m & . & . & \\ 4 & f & . & . & . & c \\ \hline & 6 & 7 & 8 & 9 & 3 \end{array} = \underline{|T_r(k+1)}$$

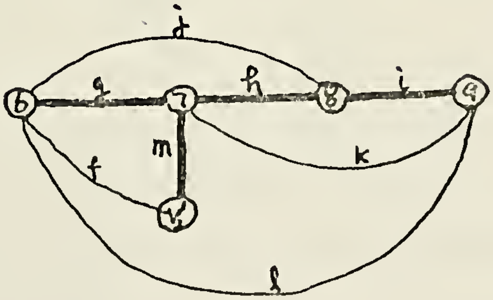
Note:
Rows and columns of $\underline{|T(k+1)}$ in Step 1 are rearranged according to specified circuit C(k+1).

$$\underline{|T_r'(k+1)} = \begin{array}{c|cccc} 7 & g & & & \\ 8 & j & h & & \\ 9 & l & k & i & \\ V_1' & f & m & . & . \\ \hline & 6 & 7 & 8 & 9 \end{array}$$

L(k+1) = f,l,k

U(k+1) = j

N(k+1) = none



According to the above test the decomposed subgraph G'(k+1) is planar.

Step 5 Implement edge T-matrices associated with the decomposed subgraph $G_1''(k+1)$. Place these edge T-matrices in the matrix list. Delete $\underline{\|T(k+1)}$ from the matrix list and return to Step 1.

$$\underline{\|T(k+3)} = \begin{array}{c|ccccc} 7 & g & & & & \\ 8 & . & h & & & \\ 9 & l & . & i & & \\ 3 & . & m & . & . & \\ 4 & f & . & . & . & c \\ \hline & 6 & 7 & 8 & 9 & 3 \end{array}$$

For the k+3th iteration

The k+2th iteration is skipped to show the relation between the k+1th iteration and k+3th iteration.

Step 1 It is assumed the $T(k+3)$ is selected

$$\underline{\|T(k+3)} = \begin{array}{c|ccccc} 7 & g & & & & \\ 8 & . & h & & & \\ 9 & l & . & i & & \\ 3 & . & m & . & . & \\ 4 & f & . & . & . & c \\ \hline & 6 & 7 & 8 & 9 & 3 \end{array} = \underline{\|T_r(k+3)}$$

Step 2 Completed in Step 1.

Step 3 The branches of the resolving tree $\overset{T_r(k+3)}{_\wedge}T_r(k+3)$ obtained from $\underline{|T_r(k+3)}$ are:

$$T_r(k+3) = (g,h,i,m,c)$$

The chords specified with respect ot $T_r(k+3)$ are:

Class-1 chords: l
Class-2 chords: none
Class-3 chords: f

There is only one Class-3 chord "f" so the $C(k+3)$ is the fundamental circuit specified by chord "f"

$$C(k+3) = \underline{f}, g, m, c$$

As it is seen above, $C(k+3)$ is equal to $C(k)$, so the algorithm does not converge. Planarity test is failed in this case.

Example 7 Consider the graph given in Example 6. The edge T-matrix given below is associated with the graph $G(k)$ in Example 6. Using this edge T-matrix, the planarity of $G(k)$ is tested.

$$\underline{\left| T(k) \right.} = \begin{array}{c|cccccccc} 2 & a \\ 3 & . & b \\ 7 & . & . & m \\ 8 & . & . & . & h \\ 9 & . & . & . & k & i \\ 6 & . & . & . & g & j & l \\ 4 & . & . & c & . & . & . & f \\ 5 & e & . & . & . & . & . & . & d \\ \hline & 1 & 2 & 3 & 7 & 8 & 9 & 6 & 4 \end{array} = \underline{\left| T_r(k) \right.}$$

For the kth iteration

Step 1 $\underline{|T(k)}$ given above is selected from the matrix list.

Step 2 Completed in the given $\underline{|T(k)}$.

Step 3 The branches of the resolving tree $T_r(k)$ obtained from $\underline{|T_r(k)}$ are:

$$T_r(k) = (a,b,m,h,i,l,f,d)$$

The chords specified with respect to $T_r(k)$ are:

Class-1 chords: k,j,g,c,e

Class-2 chords: none

Class-3 chords: none

The fundamental circuits obtained according to the chords with respect to the specified resolving tree $T_r(k)$ are:

$C_1 = \underline{k},i,h$

$C_2 = \underline{j},l,i$

$C_3 = \underline{g},l,i,h$

$C_4 = \underline{c},f,l,i,h,m$

$C_5 = \underline{e},d,f,l,i,h,m,b,a$

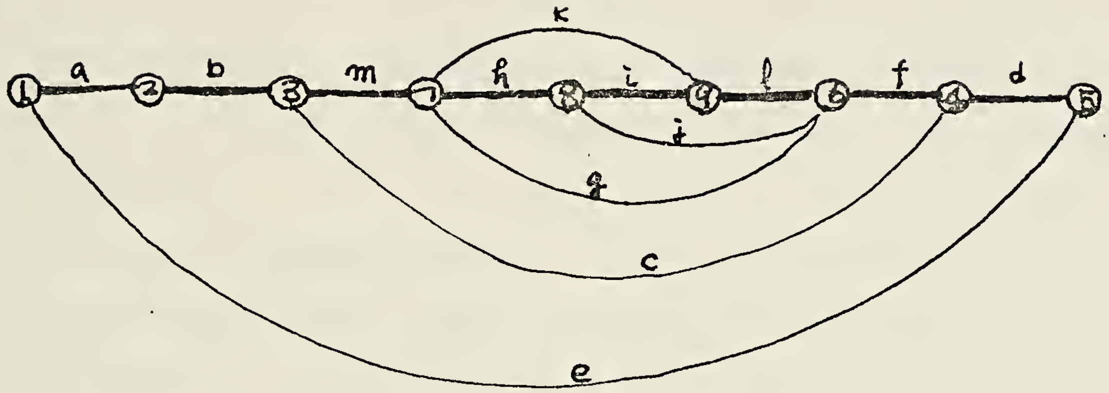
C_5 has the greatest number of edges among the above fundamental circuits and so C_5 is selected as circuit C(k).

Step 4 Implement the edge T-matrix $\underline{|T'(k)}$ associated with G'(k) and test the planarity of G'(k). In this case specified circuit C(k) is the Hamilton circuit and so, $\underline{|T(k)} = \underline{|T'(k)}$ ..

L(k) = e,c,g,j

U(k) = k

N(k) = none



According to the above test the decomposed subgraph G'(k) is planar. So the G(k) is planar because T'(k) is the same as T(k).

B. THE PLANARITY OF A PSEUDO-HAMILTONIAN GRAPH

Planarity test and planar partition of a graph given in the above examples were done by using the Phung-Chan method. As it was seen in the above examples Theorem 5, the necessary

and sufficient condition that a pseudo-Hamiltonian graph be planar, did not work. The graphs in Example 1 and Example 2 are isomorphic and the graphs given in Examples 3, 4 and 5 are the same and they are all pseudo-Hamiltonian graphs with respect to the specified circuits. However, the results obtained by the application of Theorem 5 to Examples 1 and 2 are different from each other and the results obtained by the application of Theorem 5 to the Examples 3 and 5 are the same, but they are different from the result obtained by the application of Theorem 5 to Example 4. As mentioned above, the application of Theorem 5 depends on the specified pseudo-Euler tree. If the pseudo-Euler tree is properly specified, Theorem 5 works. If the pseudo-Euler tree is not properly specified, there are chances that Theorem 5 can not be applied to the problems and it is very difficult to find a proper pseudo-Euler tree by inspection when a graph is complex.

In the next section an algorithm will be given for determining a proper pseudo-Euler tree. Then the Phung-Chan algorithm for testing the planarity of a pseudo-Hamiltonian graph together with this algorithm will be used in planarity test, planar partition and drawing of a planar subgraph of an arbitrary pseudo-Hamiltonian graph.

1. Algorithm for Determining a Proper Pseudo-Euler Tree of G'(k)

It is assumed that pseudo-Hamiltonian graph G'(k) is already obtained from decomposition of G(k) with respect

to a specified circuit C(k) and that the elements of C(k) in $\lfloor T'_r(k)$ associated with G'(k) are:

$$C(k) = t'_{11}, t'_{22}, \ldots, t'_{pp}, t'_{p1}$$

Step 1 Form a resolving edge T-matrix $\lfloor T'_r(k)$ associated with G'(k).

Step 2 Test if the element $t'_{p+1,p+1}$ of $T'_r(k)$ is zero. If $t'_{p+1,p+1}$ is zero, then the proper pseudo-Euler tree is the pseudo-Euler tree obtained from $\lfloor T'_r(k)$. If $t'_{p+1,p+1}$ is not zero, then there may be chances that Theorem 5 does not apply. Then go to Step 3.

Step 3 Form another resolving edge T-matrix $\lfloor T''_r(k)$ associated with G'(k) with the element of $\lfloor T''_r(k)$ $t''_{11} = t'_{22}$. Let $\lfloor T'_r(k) = \lfloor T''_r(k)$ and return to Step 2. Where t'_{ij} and t''_{ij} are the elements of the resolving edge T-matrices $\lfloor T'_r(k)$ and $\lfloor T''_r(k)$ respectively.

2. Partition of Chord-Sets

The partition of the set of chords, with respect to a pseudo-Euler tree of the graph, can be determined by a simple inductive procedure. The procedure follows directly from the observation that chords, i and j which alternate, are forced to be on opposite sides of the trunk of the pseudo-Euler tree. Using this observation we place a Class-1 chord or a set of Class-2 chords which are incident at a same attachment tree branch in "L" or "U" if and only if this Class-1 chord or a set of Class-2 chords alternate with chords which have already been so placed.

Before partitioning the chords some notations are defined. Let H(k) be the row matrix whose elements are set of chords defined with respect to the pseudo-Euler tree $T_r'(k)$ of $T_r'(k)$ and H' be the row matrix whose elements are chords in H(k) which have not been placed. Initially H' = H(k).

To start the procedure we select an arbitrary Class-1 chord or a set of Class-2 chords which are incident at a same attachment tree branch in H' and denote this as submatrix H_o of H(k), and then put H_o in "L" or "U". Next we find all of the Class-1 chords or sets of Class-2 chords in H' that alternate with H_o and denote this as submatrix H_1. If any two chords in H_1 alternate, the partitioning fails and the pseudo-Hamiltonian Graph G'(k) is nonplanar.

Assume next that a partitioning has been obtained up to a set of submatrices H_n so that H(k) is given by

$$H(k) = [H_o \quad H_1 \quad H_2 \quad \cdot \cdot \cdot \cdot \cdot \quad H_n \quad H']$$

The submatrix H_{n+1} is formed by determining all of the Class-1 chords and sets of Class-2 chords in H' which alternate with submatrix H_n. The submatrix H_{n+1} is associated with the side of the trunk of the pseudo-Euler tree $T_r'(k)$ opposite to that associated with H_n and the procedure is repeated. It may happen that there are no chords in H' which alternate with chords in H_n. In this case an arbitrary Class-1 chord or set of Class-2 chords which are incident at a same attachment tree branch in H' is selected as H_o' and

the procedure is repeated. In this more general case the partitioning of H(k) takes the form

$$H(k) = [H_o H_1 \cdots H_k H'_o H'_1 \cdots H'_p H''_o H''_1 \cdots]$$

in which $H_o, H_1, \ldots, H_k$ are alternately placed in "L" or "U", similarly for $H'_o, H'_1, \ldots, H'_p$, etc.

3. Identification of a Planar Subgraph of a Nonplanar Pseudo-Hamiltonian Graph

If a pseudo-Hamiltonian graph G(k) is nonplanar, the technique for partitioning the row matrix H(k) can be modified by deleting chords to result in the identification of a planar subgraph of G'(k).This modification, when employed with the iterative algorithm, also enables the identification of a planar subgraph of an arbitrary graph.

The procedure for deleting chords to resume the partitioning is straightforward. Denote by A' a set of chords in H_n which cannot be placed, and by A a set of placed chords in H_n with which chords in A' are alternate. If A' contains Class-1 chords, these chords are nonplanar and are deleted. If A' also contains sets of Class-2 chords, we delete a sufficient number of chords from each set of Class-2 chords which are incident at a same attachment tree branch so that these sets of Class-2 chords no longer alternate with any placed chords in A. Note that after deleting chords from each set of Class-2 chords in A' we must check to insure that each set still alternates with

at least one chord in H_{n-1}. Otherwise, the sets are no longer constrained in H_n and are returned to H'.

C. IDENTIFICATION OF A PLANAR SUBGRAPH OF AN ARBITRARY GRAPH

The algorithms for the identification of a planar subgraph of an arbitrary graph reviewed in Chapter II use the decomposition theorem [9]. So the determination of C(k) having the greater number of edges as possible is most important, because, in such case, fewer decomposed subgraphs are generated. The algorithms in Chapter II converge most rapidly when the decomposition with respect to C(k) is pseudo-Hamiltonian since in this case no further decomposed subgraphs are generated.

Using the edge T-matrix, Chang and Chan have given an effective algorithm for listing all paths between two specified vertices of a graph and an algorithm for listing all circuits of a graph. If the Chan-Chang path listing and circuit listing algorithms are used, it is possible to get the C(k) which has the greatest number of edges among the circuits. However, unfortunately, both of the Chan-Chang path listing and circuit listing algorithms require a computation time which increases very rapidly with the number of vertices of the graph despite their improved effectiveness. In some cases the circuit which has the greatest number of edges among circuits does not contain the pre-specified edges and so this circuit cannot be used as C(k). In the case of listing all paths between two specified vertices by using

Chan-Chang path listing algorithm 2-subset of subsequent edge T-matrices derived from the original edge T-matrix associated with G(k) are required and to get 2-subset of subsequent edge T-matrices of the original edge T-matrix associated with the graph whose number of vertices is v, (v-3)! calculations are needed and so to use either of these Chan-Chang algorithms to specify a circuit C(k) is not realistic.

In the Phung-Chan algorithm C(k) is selected from the fundamental circuits with respect to the resolving tree $T_r(k)$ of $\underline{T_r(k)}$ associated with G(k). The C(k) specified by using Phung-Chan method may not be the circuit which has the greatest number of edges among circuits, but in many cases this specfied fundamental circuit C(k) has comparatively many edges and it is very easy to get. In the case of finding C(k) of G(k), if G(k) is not separable between C(k-m) and a Type-3 bridge $B_i(k-m)$, C(k) is selected as the fundamental circuit which has the greatest number of edges among the fundamental circuits obtained according to chords of attachment edges. In some cases as it is seen in Example 6, the specified circuit C(k) does not contain many edges, so the rate of convergence of the algorithm is very slow or the algorithm does not converge. As it was pointed out in the above, there is need to develop other methods to find a proper circuit C(k) in order that the algorithm may converge comparatively rapidly. Another algorithm for finding a proper circuit C(k) is suggested in the following.

1. Algorithm for Finding C(k) of G(k)

It is assumed that G(k) is composed of circuit C(k-m) and Type-3 Bridge $B_i(k-m)$. If the graph to be tested is G(1), then C(1) is specified as the one which has the greatest number of edges among the fundamental circuits with respect to the resolving tree $T_r(1)$ of $\underline{|T_r(1)}$ associated with G(1)

Step 1 Form the edge T-matrix $\underline{|T(k)}$ of G(k). First part of the columns of $\underline{|T(k)}$ of G(k) are the ordered vertices of circuit C(k-m) and the second part of the columns are the vertices of $G_i(k-m)$. Forming the edge T-matrix $\underline{|T(k)}$ of G(k), do not change the sequence of vertices of $G_i(k-m)$. Doing this, $\underline{|T(k)}$ of G(k) is always resolving edge T-matrix.

Step 2 Test if the decomposed subgraph G(k) is separable between C(k-m) and $B_i(k-m)$, or between C(k-m) with attachment edges and $G_i(k-m)$. If G(k) is separable between C(k-m) and $B_i(k-m)$, or between C(k-m) with attachement edges and $G_i(k-m)$, then C(k) is the fundamental circuit which has the greatest number of edges among the fundamental circuits with respect to the resolving tree of $B_i(k-m)$ or $G_i(k-m)$ respectively. If the decomposed subgraph G(k) is not separable, then go to Step 3. If G(k) is separable, then the planarity test will be done only to $B_i(k-m)$ or $G_i(k-m)$.

Step 3 Delete the vertices of C(k-m) at which the attachment edges are not incident and delete the edges which are incident at these vertices from the $\underline{|T(k)}$ of G(k). Then connect the

remaining vertices in C(k-m) with edge P_i consecutively. Edge T-matrix obtained by this operation is always resolving edge T-matrix. If there appears any parallel edge P_i in the process of this operation, then delete this parallel edge P_i.

Step 4 List the whole fundamental circuits according to the chords with respect to the resolving tree which is obtained from the resolving edge T-matrix formed in step 3. Specify the circuit C(k) as the one which has the greatest number of edges among the fundamental circuits obtained according to chords.

Two algorithms given above will be illustrated in the examples in Chapter IV.

D. ALGORITHM FOR IDENTIFYING A PLANAR SUBGRAPH FROM AN ARBITRARY GRAPH

Phung-Chan algorithm is modified. The input is the edge T-matrix $\underline{|T(1)}$ of an arbitrary graph G(1). The algorithm identifies planar subgraphs of G(1) This algorithm is initialized by placing $\underline{|T(1)}$ in the matrix list.

Step 1 Test if there is an edge T-matrix in the matrix list. If not, the run is over.

Step 2 Transform $\underline{|T(k)}$ into resolving edge T-matrix $\underline{|T_r(k)}$.

Step 3 Using the algorithm for finding C(k) of G(k) given in Section B.1 in this chapter, specify C(k). If G(k) is separable between C(k-m) and Type-3 bridge B_i(k-m)(or between C(k-m) with attachment edges and connected component

$G_i(k-m)$), let $G(k)$ be $B_i(k-m)$ (or $G_i(k-m)$) and let $\lfloor T(k)$ be the edge T-matrix of $B_i(k-m)$ (or $G_i(k-m)$). If no circuit $C(k)$ can be formed, $G(k)$ is planar. Return to Step 1.

Step 4 Implement the edge T-matrix, $\lfloor T'(k)$ associated with decomposed subgraph $G'(k)$. Using Theorem 5 together with the algorithm for determining a proper pseudo-Euler tree of $G'(k)$, test the planarity of $G'(k)$. If we are only interested in testing planarity, the procedure terminates at this point. Otherwise, place nonplanar edges in $N(k)$.

Step 5 If the decomposition of $G(k)$ with respect to $C(k)$ is pseudo-Hamiltonian, delete $\lfloor T(k)$ from the matrix list and then return to Step 1. Otherwise, implement edge T-matrices associated with each Type-3 bridge $B_i(k)$ of $C(k)$. Place these edge T-matrices in the matrix list. Delete $\lfloor T(k)$ from the matrix list and return to Step 1.

Note: In performing Step 4, if there is any chord "P_i" which is generated in the process of specifying $C(k)$, put these edges first in each submatrix H_i of $H(k)$ so that $N(k)$ may not have any edge "P_i". For identifying planar subgraphs put the edge T-matrix whose elements are edges deleted from $G(1)$ in the process of partitioning $H(k)$. These deleted edges are

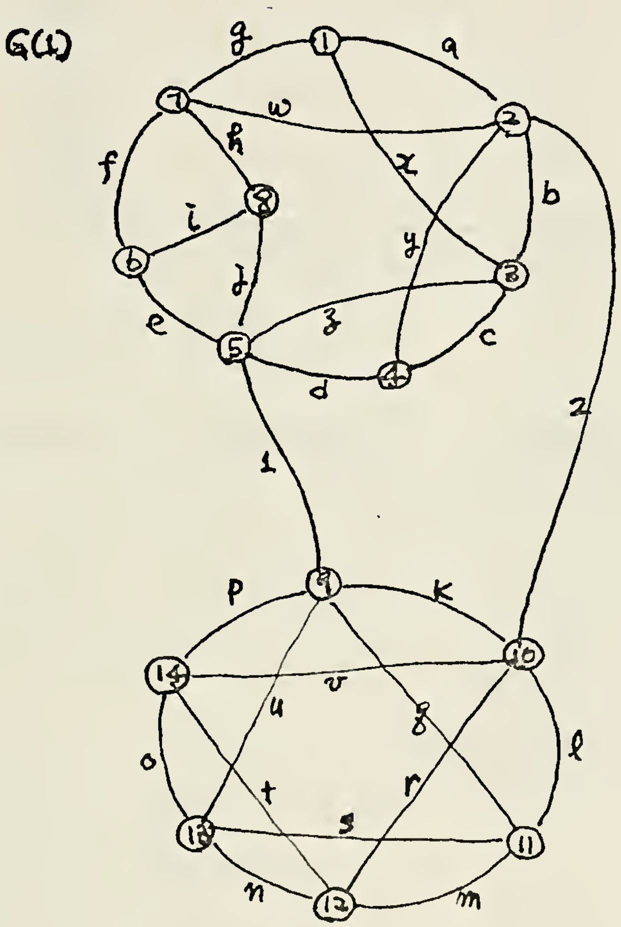
$$\sum_{k=1}^{n} N(k) \quad .$$

The iteration will be finished when there appear no deleted edges.

IV. APPLICATION OF ALGORITHM

Planarity test and planar partitioning of a graph will be done in the following examples by using the algorithm given in Chapter III. By these examples the effectiveness of the algorithm will be shown.

Example 8 The planarity of the graph G(1) given below is tested by using the algorithm given in Chapter III.



For the first run

Step 1

$$\underline{\vert T(1)} = \begin{array}{r|ccccccccccccc}
2 & a \\
3 & x & b \\
4 & . & y & c \\
5 & . & . & z & d \\
6 & . & . & . & . & e \\
7 & g & w & . & . & . & f \\
8 & . & . & . & . & j & i & h \\
9 & . & . & . & . & 1 & . & . & . \\
10 & . & 2 & . & . & . & . & . & . & k \\
11 & . & . & . & . & . & . & . & . & q & l \\
12 & . & . & . & . & . & . & . & . & . & r & m \\
13 & . & . & . & . & . & . & . & . & u & . & s & n \\
14 & . & . & . & . & . & . & . & . & p & v & . & t & o \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13
\end{array} = \underline{\vert T_r(1)}$$

Step 2 Completed in Step 1.

Step 3 From the resolving edge T-matrix $\underline{\vert T_r(1)}$ resolving tree $T_r(1)$ is obtained. The branches of $T_r(1)$ are:

$$T_r(1) = (a,b,c,d,e,f,h,i,k,l,m,n,o)$$

The chords defined with respect to $T_r(1)$ are:

Class-1 chords: x,y,z,g,w,j,i,q,r,s,t,u,v,p

Class-2 chords: none

Class-3 chords: 2.

The fundamental circuit which has the greatest number of edges among fundamental circuits according to the specified chords with respect to the resolving tree $T_r(1)$ is:

$C_g = \underline{g},a,b,c,d,e,f$

C_g is selected as circuit C(1).

Step 4 Implement the edge T-matrix associated with G'(1) and test the planarity of G'(1).

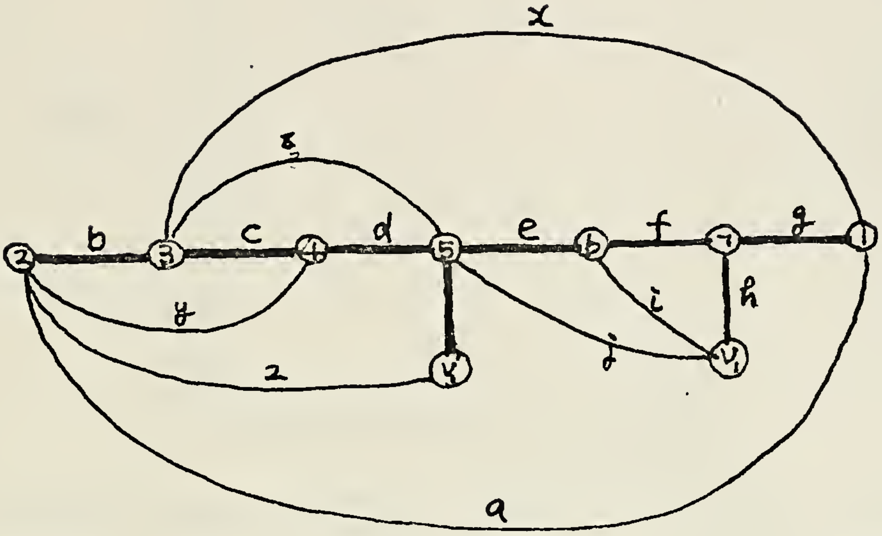
$$\lfloor\underline{T'(1)} = \begin{array}{c|cccccccc}
2 & a \\
3 & x & b \\
4 & . & y & c \\
5 & . & . & z & d \\
6 & . & . & . & . & e \\
7 & g & . & . & . & . & f \\
V_1 & . & . & . & . & j & i & h \\
V_1' & . & 2 & . & . & 1 & . & . & . \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & V_1
\end{array}$$

The circuit C(1) has seven vertices and the entry $t_{7,7}$ (= "h") of $\lfloor\underline{T'(1)}$ associated with decomposed subgraph G'(1) is not zero. And so reform the edge T-matrix $\lfloor\underline{T'(1)}$ with t_{11} of $\lfloor\underline{T'(1)}$ as "b".

$$\lfloor\underline{T'(1)} = \begin{array}{c|cccccccc}
3 & b \\
4 & y & c \\
5 & . & z & d \\
6 & . & . & . & e \\
7 & . & . & . & . & f \\
1 & a & x & . & . & . & g \\
V_1 & . & . & . & j & i & h & . \\
V_1' & 2 & . & . & 1 & . & . & . & . \\
\hline
 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & V_1
\end{array}$$

$$H(1) = [\overset{L}{a} \vdots \overset{L}{y} \vdots \overset{U}{z,x} \vdots \overset{L}{y,2} \vdots \overset{U}{j\ i}]$$

$$N(1) = [\ \text{zero}\ ]$$



Note: pseudo-Euler tree obtained from T'(1) is shown as a darkened line.

According to the above test the decomposed subgraph G'(1) is planar.

Step 5

$$\lfloor T(2) = \begin{array}{r|cccccccccccc}
2 & a & & & & & & & & & & & \\
3 & . & b & & & & & & & & & & \\
4 & . & . & c & & & & & & & & & \\
5 & . & . & . & d & & & & & & & & \\
6 & . & . & . & . & e & & & & & & & \\
7 & g & . & . & . & . & f & & & & & & \\
9 & . & . & . & . & 1 & . & . & & & & & \\
10 & . & 2 & . & . & . & . & . & k & & & & \\
11 & . & . & . & . & . & . & . & q & l & & & \\
12 & . & . & . & . & . & . & . & . & r & m & & \\
13 & . & . & . & . & . & . & . & u & . & s & n & \\
14 & . & . & . & . & . & . & . & p & v & . & t & o \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12
\end{array} = \lfloor T_r(2)$$

The edge T-matrix $\lfloor T(2)$, associated with the decomposed subgraph G(2), is composed of C(1) and the the subgraph $G_1(1)$ with vertices "9", "10", "11", "12", "13", "14".

For the second run

Step 1 T(2) is selected.

Step 2 Completed in Step 5 in first run.

Step 3 Reform the above edge T-matrix $\lfloor T_r(2)$ to $\lfloor T_r(2')$ according to the above suggested algorithm for finding circuit C(k).

$$\lfloor\underline{T_r(2')} = \begin{array}{r|ccccccc} 5 & P_1 & & & & & & \\ 9 & . & l & & & & & \\ 10 & 2 & . & k & & & & \\ 11 & . & . & q & l & & & \\ 12 & . & . & . & r & m & & \\ 13 & . & . & u & . & s & n & \\ 14 & . & . & p & v & . & t & o \\ \hline & 2 & 5 & 9 & 10 & 11 & 12 & 13 \end{array}$$

From the above resolving edge T-matrix $\lfloor\underline{T_r(2')}$ resolving tree $T_r(2')$ is obtained. The branches of $T_r(2')$ are:

$$T_r(2') = (P_1, l, k, l, m, n, o)$$

The chords defined with respect to $T_r(2')$ are:

Class-1 chords: 2,q,r,s,t,u,v,p

Class-2 chords: none

Class-3 chords: none

Note: Class-2 and Class-3 chords are none.

If decomposed subgraph G(k) which is composed of C(k-m) and $G_i(k-m)$ is not separable from each other, then associated edge T-matrix $\lfloor\underline{T_r(k')}$ has always class-1 chords.

The fundamental circuit which has the greatest number of edges among fundamental circuits according to the specified chords with respect to the resolving tree $T_r(2')$ is:

$$C_p = \underline{p}, k, l, m, n, o$$

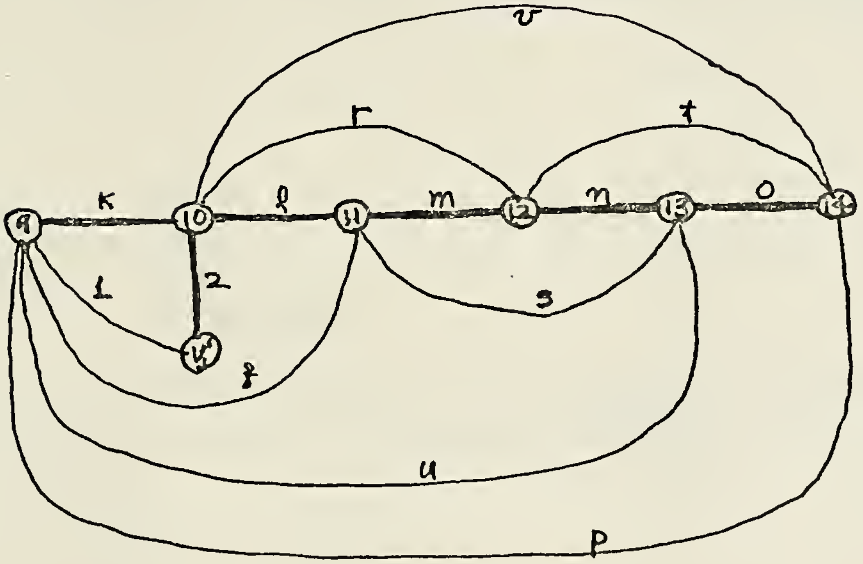
C_p is selected as the circuit C(2).

Step 4 Implement the edge T-matrix associated with G'(2) and test the planarity of G'(2)

$$
T'(2) = \begin{array}{c|cccccc}
10 & k & & & & & \\
11 & q & l & & & & \\
12 & . & r & m & & & \\
13 & u & . & s & n & & \\
14 & p & v & . & t & o & \\
V'_1 & 1 & 2 & . & . & . & . \\
\hline
 & 9 & 10 & 11 & 12 & 13 & 14
\end{array}
$$

$$H(2) = [\overset{L}{p} \,\vdots\, \overset{L}{q} \,\vdots\, \overset{U}{r,v} \,\vdots\, \overset{L}{s} \,\vdots\, \overset{U}{t} \,\vdots\, \overset{L}{u}]$$

N(2) = [none]



According to the above test the decomposed subgraph G'(2) is planar.

Step 5

$$\lfloor\underline{T(3)} = \begin{array}{r|ccccccc}
10 & k & & & & & & \\
11 & . & 1 & & & & & \\
12 & . & . & m & & & & \\
13 & . & . & . & n & & & \\
14 & p & . & . & . & o & & \\
2 & . & 2 & . & . & . & . & \\
5 & 1 & . & . & . & . & . & p_1 \\
\hline
 & 9 & 10 & 11 & 12 & 13 & 12 & 2
\end{array} = \lfloor\underline{T_r(3)}$$

For the third run

Step 1 T(3) is selected from the matrix list.

Step 2 Completed in Step 5 in the second run.

Step 3 Reform the edge T-matrix $\lfloor\underline{T_r(3)}$ to $\lfloor\underline{T_r(3')}$ by using the algorithm for finding circuit C(k) given above.

$$\lfloor\underline{T_r(3')} = \begin{array}{r|ccc}
10 & p_2 & & \\
2 & . & 2 & \\
5 & 1 & . & p_1 \\
\hline
 & 9 & 10 & 2
\end{array}$$

From the resolving edge T-matrix $\lfloor\underline{T_r(3')}$ resolving tree $T_r(3')$ is obtained. The branches of $T_r(3')$ are:

$$T_r(3') = (p_2, 2, p_1)$$

The chords defined with respect to $T_r(3')$ are:

Class-1 chords: 1

The fundamental circuit C(3) is C_1 because there is only one chord "1" with respect to the resolving tree $T_r(3')$. The branches of C_1 are:

$$C_1 = \underline{1}, p_2, 2, p_1$$

Step 4 Implement the edge T-matrix associated with G'(3) and test the planarity of G'(3).

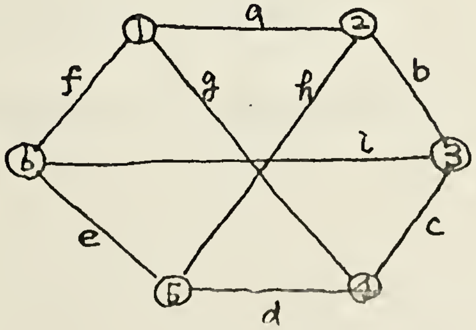
$$\underline{\left| T'(3) \right.} = \begin{array}{c|ccc} 10 & p_2 & & \\ 2 & . & 2 & \\ 5 & 1 & . & p_1 \\ \hline & 9 & 10 & 2 \end{array} = \underline{\left| T(3') \right.}$$

H(3) = [1]

N(3) = [none]

G'(3) is planar. And so the given graph G(1) is planar. Iteration is finished.

Example 9 In this example we will test the planarity of the Kuratowsty's one of nonplanar graphs by using the algorithm given in Chapter III.



For the first run

Step 1

$$\underline{\left|T(1)\right.} = \begin{array}{c|ccccc} 2 & a \\ 3 & . & b \\ 4 & g & . & c \\ 5 & . & h & . & d \\ 6 & f & . & i & . & e \\ \hline & 1 & 2 & 3 & 4 & 5 \end{array} = \underline{\left|T_r(1)\right.}$$

Step 2 Completed in Step 1.

Step 3 The branches of resolving tree $T_r(1)$ of $\underline{\left|T_r(1)\right.}$ are:

$$T_r(1) = (a,b,c,d,e)$$

The chords defined with respect to $T_r(1)$ are:

Class-1 chords: g,h,i,f

Class-2 chords: none

Class-3 chords: none

The fundamental circuit which has the greatest number of edges among fundamental circuits according to the specified chords with respect to the resolving tree $T_r(1)$ is:

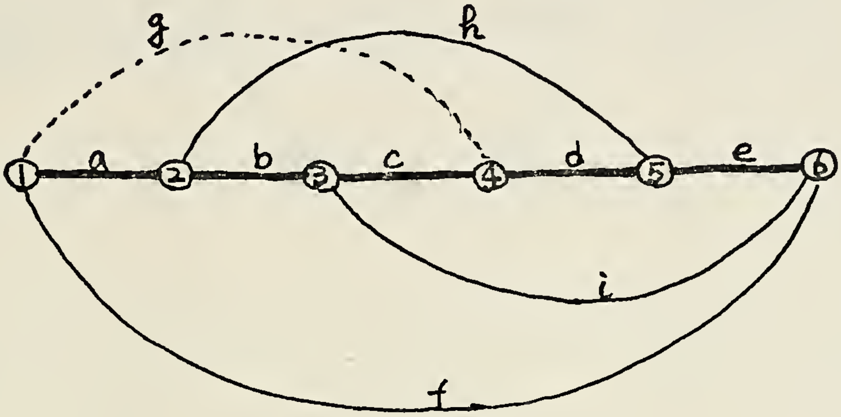
$$C_f = \underline{f},a,b,c,d,e$$

C_f is selected as circuit C(1).

Step 4 Implement the edge T-matrix associated with G'(1) and test the planarity of G'(1). In this example G'(1) with respect to C(1) is the same as G(1).

$$H(1) = \begin{array}{ccccc} L & & L & & U \\ [f & \vdots & i & \vdots & h] \end{array}$$

$$N(1) = [g]$$

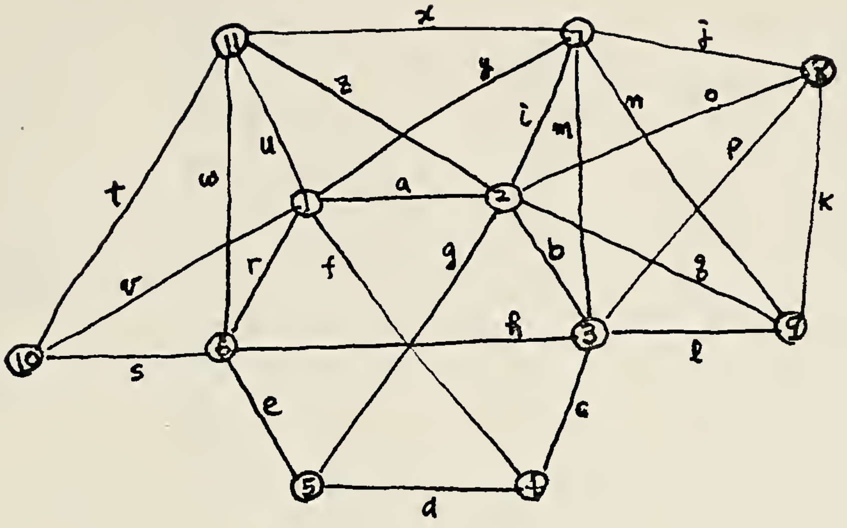


Note: Nonplanar edge "g" is drawn as a dotted line and pseudo-Euler tree $T_r(1)$ is drawn as a darkened line.

According to Step 4 test, G(1) is nonplanar.

Example 10 The algorithm given in Chapter III will be illustrated for the planarity test and planar partitioning of an arbitrary graph given below.

G(1)



For the first run

Step 1

$$\underline{T(1)} = \begin{array}{r|cccccccccc}
2 & a \\
3 & . & b \\
4 & f & . & c \\
5 & . & g & . & d \\
6 & r & . & h & . & e \\
7 & y & i & m & . & . & . \\
8 & . & o & p & . & . & . & j \\
9 & . & q & l & . & . & . & n & k \\
10 & v & . & . & . & . & s & . & . & . \\
11 & u & z & . & . & . & w & x & . & . & t \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}$$

Step 2

$$\underline{\lfloor T_r(1)} = \begin{array}{r|cccccccccc}
2 & a \\
3 & . & b \\
4 & f & . & c \\
5 & . & g & . & d \\
6 & r & . & h & . & e \\
10 & v & . & . & . & . & s \\
11 & u & z & . & . & . & w & t \\
7 & y & i & m & . & . & . & . & x \\
8 & . & o & p & . & . & . & . & . & j \\
9 & . & q & l & . & . & . & . & . & n & k \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 10 & 11 & 7 & 8
\end{array}$$

Step 3 The branches of resolving tree $T_r(1)$ of $\underline{\lfloor T_r(1)}$ are:

$T_r(1) = (a,b,c,d,e,s,t,x,j,k)$

The chords defined with respect to $T_r(1)$ are:

Class-1 chords: f,g,h,w,n,r,v,z,m,u,i,p,q,o,l,a

Class-2 chords: none

Class-3 chords: none

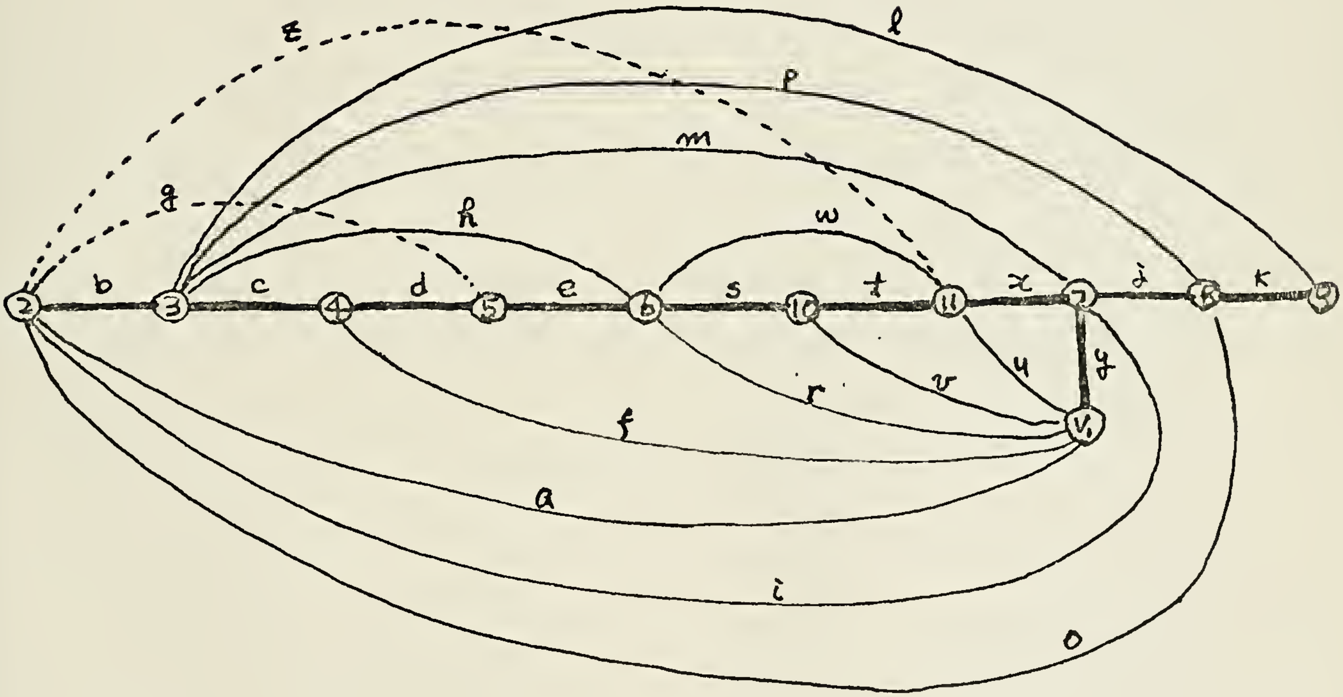
The fundamental circuit which has the greatest number of edges among fundamental circuits according to the specified chords with respect to the resolving tree $T_r(1)$ is

$C_g = \underline{g}, b,c,d,e,s,t,x,j,k$

C_g is selected as circuit C(1).

Step 4

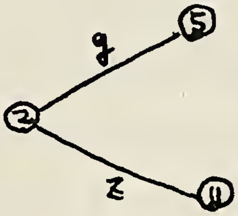
$$T'(1) = \begin{array}{c|cccccccccc}
3 & b \\
4 & . & c \\
5 & g & . & d \\
6 & . & h & . & e \\
10 & . & . & . & . & s \\
11 & z & . & . & . & w & t \\
7 & i & m & . & . & . & . & x \\
8 & o & p & . & . & . & . & . & j \\
9 & q & l & . & . & . & . & . & n & k \\
V_1 & a & . & f & . & r & v & u & y & . & . \\
\hline
 & 2 & 3 & 4 & 5 & 6 & 10 & 11 & 7 & 8 & 9
\end{array}$$



$$H(k) = [a\ f\ \overset{L}{r}\ v\ u \,\vdots\, m,\overset{L}{p},l,w \,\vdots\, \overset{L}{n},o \,\vdots\, \overset{L}{q}\ ]$$

$N(k) = [z,g]$

According to Step 4 test G(1) is nonplanar. Planar subgraph of G(1) is drawn with continuous line. Another subgraph is given below.



VI. DISCUSSION

1. In Chapter II we reviewed three algorithms. As it is seen in Chapter II, basic principles of the three algorithms are exactly the same. As they all use the decomposition theorem [9] in their algorithms, these three algorithms strongly depend on the determination of the circuit $C(k)$. The algorithms converge most rapidly when the decomposition with respect to $C(k)$ is pseudo-Hamiltonian, since in this case no further decomposed subgraphs are generated.

Using the edge T-matrix Chan and change have given an effective algorithm for listing all paths between two specified vertices of a graph and also have given an algorithm for listing all circuits of a graph. Using Chan-Chang path listing algorithms or circuit listing algorithm, circuit $C(k)$ which has the greatest number of edges among circuits can be obtained. But these algorithms require a computation time which increases very rapidly with the number of vertices of the graph despite their improved effectiveness. In Phung-Chan algorithm $C(k)$ is selected from the fundamental circuits with respect to the resolving tree $T_r(k)$ of $\underline{T_r(k)}$ associated with $G(k)$. The $C(k)$ specified by using Phung-Chan method may not be the circuit which has the greatest number of edges among circuits but in many cases this specified circuit $C(k)$ has comparatively many edges. However, as it is seen in

Example 6, the circuit C(k) sometimes does not contain many edges, so the rate of convergence of the algorithm is very slow or the algorithm does not converge.

In this paper Phung-Chan algorithm is improved by modifying the circuit finding algorithm and by establishing the algorithm for determining a proper pseudo-Euler tree of G'(k). The algorithm given in this paper converges comparatively rapidly and has not any chance that the algorithm does not converge as it is seen in Chapter IV.

2. In digital computation the algorithm is simple in terms of arithmetic and logic operation, but the computer storage requirement becomes prohibitive when the graph is large. As the algorithm given in this paper also uses edge T-matrix, the computer storage requirement can be minimized. The computer storage requirement of edge T-matrix is smaller than half of that of incidence matrix of the same graph which Fisher and Wing used in their algorithm.

3. The algorithm established in this paper is comparatively easy and simple and so hand calculation is possible.

VI. CONCLUSION

In this paper the Phung-Chan algorithm is improved by establishing the algorithm for determining a pseudo-Euler tree and by modifying circuit finding algorithm. By using this algorithm the rate of convergence is increased and the computer storage is minimized.

LIST OF REFERENCES

1. C. Kuratowski, "Sur le probleme des courbes gauches en topologie," Fundamenta Math., vol. 15, pp. 271-83, 1930.

2. H. Whitney, "Non-separable and planer graphs," Trans. Amer. Math. Soc., vol. 34, pp. 339-62, April 1932.

3. L. Auslander and S.V. Porter, "On imbedding graphs in the sphere," J. Math. Mech., vol. 10, pp. 517-23, May 1961.

4. A.J. Goldstein, "An efficient and constructive algorithm for testing whether a graph can be embedded in the plane," Bell Telephone Labs., Murray Hill, N.J., 1963.

5. G.J. Fisher and O. Wing, "Computer recognition and extraction of planar graphs from the incidence matrix," IEEE Trans. Circuit Theory, vol. CT-13, pp. 154-63, June 1966.

6. P.M. Lin, "On methods of detecting planar graphs," Proc. Eighth Midwest Symp. on Circuit Theory, Colorado State University, Boulder, pp. 1.1-1.9, June 1965.

7. Le Phung and S.G. Chan, "On the determination of Planar graphs," Proc. Third Asiloma Conference on Circuits & Systems, pp. 467-471, December 1969.

8. Le Phung and S.G. Chan, "Edge t-matrix in network theory," Thesis, Naval Postgraduate School, 1970.

9. W.T. Tutte, "A theorem on Planar Graphs," Trans. Amer. Math. Soc., vol. 82, pp. 99-116, May 1956.

10. Le Phung and S.G. Chan, "Application of foldout in graph analysis," Proc. Third Hawaii International Conference on Systems Science, 1970.

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